

# $C^*$ -CROSSED PRODUCTS AND SHIFT SPACES

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**Abstract.** In this article, we use Exel's construction to associate a  $C^*$ -algebra to every shift space. We show that it has the  $C^*$ -algebra defined in [13] as a quotient, and possesses properties indicating that it can be thought of as the universal  $C^*$ -algebra associated to a shift space. We also consider its representations, relationship to other  $C^*$ -algebras associated to shift spaces, show that it can be viewed as a generalization of the universal Cuntz-Krieger algebra, discuss uniqueness and a faithful representation, provide conditions for it being nuclear, for satisfying the UCT, for being simple, and for being purely infinite, show that the constructed algebras and thus their K-theory,  $K_0$  and  $K_1$ , are conjugacy invariants of one-sided shift spaces, present formulas for those invariants, and also present a description of the structure of gauge invariant ideals.

**Keywords:**  $C^*$ -algebra, shift spaces, dynamical systems, invariants.

## 1. INTRODUCTION

When dynamical system consists of a homeomorphism of a topological space, or more generally when an action of a group of invertible transformations of some space is studied, there is a standard construction of a crossed product  $C^*$ -algebra. Historically this construction has its origins in foundations of quantum mechanics. The important idea behind this construction is that it encodes the action and the space within one algebra thus providing opportunities for their investigation on the same level. It is known that properties of the topological space can be considered via properties of the algebra of continuous functions defined on it. The crossed product algebra is constructed by combining this algebra of functions with the action being encoded using further elements of the new in general non-commutative algebra. The action is built into multiplication in the

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new algebra via covariance commutation relations between the elements in the algebra of functions and the elements used to encode the action. The crossed product construction have considerable applications in quantum mechanics and quantum field theory, and provide an important source of examples for further development of non-commutative geometry. A lot of research has been done on interplay between properties of the invertible dynamical systems and properties of the corresponding crossed product  $C^*$ -algebras and  $W^*$ -algebras.

There are several ways to generalize the construction of the  $C^*$ -crossed product to the non-invertible setting. The one we will focus on in this paper was introduced by Exel in [16]. This construction relies on a choice of *transfer operator*. Exel showed that for a natural choice of transfer operator, the  $C^*$ -algebra of a one-sided shift of finite type is isomorphic a *Cuntz-Krieger algebra*.

The Cuntz-Krieger algebras was introduced by Cuntz and Krieger in [15]. They can in a natural way be viewed as universal  $C^*$ -algebras associated with shift spaces (also called subshifts) of finite type. From the point of view of operator algebra these  $C^*$ -algebras were important examples of  $C^*$ -algebras with new properties and from the point of view of topological dynamics these  $C^*$ -algebras (or rather, the  $K$ -theory of these  $C^*$ -algebras) gave new invariants of shift spaces of finite type.

In [23] Matsumoto tried to generalize this idea by constructing  $C^*$ -algebras associated with every shift space and he studied them in [24, 25, 28–30]. Unfortunately there is a mistake in [28] which makes many of the results in [24, 25, 28–30] invalid for the  $C^*$ -algebra constructed in [23], and since this mistake was discovered, there has been some confusion about the right definition of the  $C^*$ -algebra associated to a shift space.

In this paper we will use Exel's construction to associate a  $C^*$ -algebra to every shift space, and we will show that it has the properties Matsumoto thought his algebra had, and thus that it satisfies all the results of [23–25, 28–30] and has the  $C^*$ -algebra defined in [13] as a quotient. Thus it seems right to think of this  $C^*$ -algebra as the universal  $C^*$ -algebra associated to a shift space.

Matsumoto's original construction associated a  $C^*$ -algebra to every *two-sided* shift space, but it seems more natural to work with *one-sided* shift spaces, so we will do that in this paper, but since every two-sided shift space comes with a canonical one-sided shift space (see below), the  $C^*$ -algebras we define in this paper can in a natural way also be seen as  $C^*$ -algebras associated to two-sided shift spaces.

## 2. $C^*$ -ALGEBRAS OF INVERTIBLE DYNAMICAL SYSTEMS

In this section we review the construction and some properties of a  $C^*$ -crossed product of a  $C^*$ -algebra by the action of the discrete group of automorphisms. In particular the invertible dynamical systems generated by homeomorphisms of topological spaces are encoded in the crossed product  $C^*$ -algebras obtained from the actions of the group of integers on the  $C^*$ -algebra of complex-valued continuous functions.

Let  $(A, G, \alpha)$  be a triple consisting of a unital  $C^*$ -algebra, discrete group  $G$  and an action  $\alpha : G \rightarrow \text{Aut}(A)$  of  $G$  on  $A$ , meaning a homomorphism from the group  $G$  into the group  $\text{Aut}(A)$  of automorphisms of the  $C^*$ -algebra  $A$ . A pair  $\{\pi, u\}$  consisting of a representation  $\pi$  of  $A$  and a unitary representation  $u$  of  $G$  on a Hilbert space  $H$  is called a covariant representation of the system  $(A, G, \alpha)$  if

$$u_s \pi(a) u_s^* = \pi(\alpha_s(a))$$

for every  $a \in A$  and  $s \in G$ . The full crossed product  $A \rtimes_\alpha G$  is defined as the universal  $C^*$ -algebra for the family of covariant representations.

Another more concrete way to define  $A \rtimes_\alpha G$  is to consider the space  $l^1(G, A)$  of all  $A$ -valued functions  $x(\cdot)$  on  $G$  with the finite  $l_1$ -norm  $\|x\| = \sum_{s \in G} \|x(s)\|_A$  equipped with the twisted convolution product and the involution

$$xy(s) = \sum_{t \in G} x(t) \alpha_t(y(t^{-1}s)), \quad x^*(s) = \alpha_s(x(s^{-1})^*)$$

making  $l^1(G, A)$  into a Banach  $*$ -algebra. The algebra  $A$  can be identified with the algebra of functions  $\tilde{a} : G \rightarrow A$  defined as  $\tilde{a}(e) = a \in A$  on the unit element  $e$  of  $G$  and as zero elsewhere on  $G$ . Moreover, for each  $s \in G$  a function  $\delta_s : G \rightarrow A$  is defined as zero everywhere on  $G$  except  $s$  where  $\delta_s(s) = 1_A$  the unit element of  $A$ . With this notation  $\tilde{a} = a\delta_e$ . It can be shown that the functions  $\delta_s$ ,  $s \in G$  are unitary elements of the Banach  $*$ -algebra  $l^1(G, A)$ , that is  $\delta_s \delta_s^* = \delta_s^* \delta_s = 1_{l^1(G, A)} = \delta_e$ ; the map  $s \mapsto \delta_s$  is a group homomorphism in the sense that  $\delta_{uv} = \delta_u \delta_v$ ; and moreover the covariance relation

$$\delta_s \tilde{a} \delta_s^* = \alpha_s(\tilde{a})$$

holds for every  $a \in A$  and  $s \in G$ . When the functions  $x \in l^1(G, A)$  are expressed as  $x = \sum_{s \in G} x(s) \delta_s$  the covariance relation implies that the operations of twisted product and involution in  $l^1(G, A)$  are the natural ones.

It can be shown that the Banach  $*$ -algebra  $l^1(G, A)$  has sufficiently many representations (i. e. for any  $a \in l^1(G, A)$  there is a representation  $\pi$  with  $\pi(a) \neq 0$ ). Thus one can define the  $C^*$ -envelope  $C^*(l^1(G, A))$  as the completion of  $l^1(G, A)$  with the norm

$$\|x\|_\infty = \sup\{\|\tilde{\pi}(x)\| \mid \tilde{\pi} \text{ is representation of } l^1(G, A)\}.$$

Any covariant representation  $\{\pi, u\}$  yields a representation  $\tilde{\pi}$  of  $l^1(G, A)$ , and hence of  $C^*(l^1(G, A))$ , defined by

$$\tilde{\pi}(x) = \sum_{s \in G} \pi(x(s)) u_s$$

for  $x$  with finite support (i. e. zero outside a finite subset of  $G$ ). Moreover, any representation of  $C^*(l^1(G, A))$  has the above form. So,  $C^*(l^1(G, A))$  is the same as the full  $C^*$ -crossed product  $A \rtimes_\alpha G$ . It is also useful to have in mind that the subspace of finite sums  $\{\sum_{s \in J} a_s \delta_s \mid J \text{ is finite, } a_s \in A\}$  is a dense  $*$ -subalgebra of  $A \rtimes_\alpha G$ .

Suppose that  $A$  is acting on a Hilbert space  $H$  and write the action as  $ah$  for  $a \in A$  and  $h \in H$ . Let  $K = l^2(G) \otimes H$  be regarded as  $l^2(G, H)$ , the space of  $H$ -valued  $l^2$ -functions on  $G$  with values in  $H$ . A pair  $\{\pi_\alpha, \lambda\}$  consisting of the representation  $\pi_\alpha$  of  $A$  and a unitary representation  $\lambda : s \mapsto \lambda_s$  of  $G$  on  $K$  defined by

$$\begin{aligned} (\pi_\alpha(a)f)(s) &= \alpha_{s^{-1}}(a)f(s), \quad f \in K, a \in A \\ (\lambda_s f)(t) &= f(s^{-1}t) \end{aligned}$$

is a covariant representation. The reduced crossed product  $A \rtimes_{\alpha, r} G$  is the  $C^*$ -algebra acting on  $K$  generated by the operator family  $\{\pi_\alpha(a), \lambda_s \mid a \in A, s \in G\}$ . It can be proved that the definition does not depend on the space  $H$ . The reduced and full crossed products are isomorphic if and only if the group  $G$  belongs to a class of so called amenable groups. In particular the group  $G = \mathbb{Z}$  of special relevance in connection to invertible dynamical systems belongs to this class.

When  $G = \mathbb{Z}$ , the number  $1 \in \mathbb{Z}$  is the generator of the group  $\mathbb{Z}$ . As  $s \mapsto \alpha_s$  is a homomorphism, it is enough to specify the defining covariance relation for  $A \rtimes_\alpha G$  for the generator of  $\mathbb{Z}$ , that is

$$\delta_1 a \delta_1^* = \alpha_1(a).$$

An object of special interest to us is the crossed product  $C^*$ -algebra for an invertible dynamical system consisting of iterations of a homeomorphism acting on a topological space.

Let  $\Sigma = (X, \sigma)$  be a topological dynamical system consisting of a homeomorphism of a Hausdorff topological space  $X$ . The  $*$ -algebra of all continuous functions on  $X$  and the  $*$ -algebra of all continuous functions on  $X$  with compact support will be denoted respectively by  $C(X)$  and by  $C_c(X)$ . The algebra  $C(X)$  has a unit if and only if  $X$  is compact, and the unit then is the constant function  $\mathbf{1} = \mathbf{1}_{C(X)}(\cdot)$  equal to 1 on all elements of  $X$ . Moreover,  $X$  is compact if and only if  $C(X)$  and  $C_c(X)$  coincide.

The mapping  $\alpha : C(X) \rightarrow C(X)$  defined by

$$(1) \quad \alpha(f)(x) = f(\sigma^{-1}(x))$$

is an automorphism of the  $*$ -algebra  $C(X)$ , and the mapping defined by

$$(2) \quad j \mapsto \alpha^j(f)(x) = f(\sigma^{-j}(x))$$

is a homomorphism of  $\mathbb{Z}$  into the group  $\text{Aut}(C(X))$  of  $*$ -automorphisms of  $C(X)$ . Since  $\sigma$  is a homeomorphism, the family of all compact subsets of  $X$  is invariant with respect to  $\sigma$  and  $\sigma^{-1}$ , and hence  $\alpha$  leaves the  $*$ -subalgebra  $C_c(X)$  of  $C(X)$  invariant. The group  $\mathbb{Z}$  is a locally compact group with respect to the discrete topology, i.e. the topology where any subset of  $\mathbb{Z}$  is open. A subset of  $\mathbb{Z}$  is compact if and only if it is finite. The set  $C_c(\mathbb{Z}, C(X))$  of continuous mappings from  $\mathbb{Z}$  to  $C(X)$  with compact support consists of all mappings which may assume non-zero values only at finitely many elements of  $\mathbb{Z}$ . For any function  $a : \mathbb{Z} \rightarrow C(X)$  we denote by  $a[k]$  the element of  $C(X)$  equal to the value of  $a$  at  $k \in \mathbb{Z}$ . The pointwise addition and multiplication by complex numbers makes  $C_c(\mathbb{Z}, C(X))$  into a linear space, which becomes a normed  $*$ -algebra with the multiplication, involution and norm defined by

$$(3) \quad (ab)[k](\cdot) = \sum_{s \in \mathbb{Z}} a[s](\cdot) \alpha^s(b[-s+k])(\cdot) =$$

$$= \sum_{s \in \mathbb{Z}} a[s](\cdot) b[-s+k](\sigma^{-s}(\cdot)),$$

$$(4) \quad b^*[k](\cdot) = \alpha^k(\overline{b[-k]})(\cdot) = \overline{b[-k]}(\sigma^{-k}(\cdot)),$$

$$(5) \quad \|b\| = \sum_{s \in \mathbb{Z}} \|b[s]\|_{C(X)}.$$

The Banach  $*$ -algebra obtained as the completion of this normed  $*$ -algebra is denoted by  $l^1(\mathbb{Z}, C(X))$ .

Let us assume that  $X$  is compact. Then  $C_c(X)$  coincides with  $C(X)$ . The  $*$ -algebra  $C(X)$  becomes a unital  $C^*$ -algebra with respect to the supremum norm defined by  $\|f\| = \|f\|_{C(X)} = \sup\{f(x) \mid x \in X\}$  for all  $f \in C(X)$ . The mappings defined by

$$\delta_j[k](\cdot) = \begin{cases} \mathbf{1} = \mathbf{1}_{C(X)}(\cdot) & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

for  $j \in \mathbb{Z}$  belong to  $C_c(\mathbb{Z}, C(X))$ , and  $\delta_0$  is the unit of  $C_c(\mathbb{Z}, C(X))$  and hence of  $l^1(\mathbb{Z}, C(X))$ . With the multiplication defined by (3), the equality  $\delta_j = \delta_1^j$  holds for all  $j \in \mathbb{Z} \setminus \{0\}$ . In what follows, for the brevity of notations, we will denote  $\delta_1$  by  $\delta$ , will assume that  $\delta^0 = \delta_0$ , and will write  $\delta^j$  instead of  $\delta_j$  for all  $j \in \mathbb{Z}$ . The algebra  $C_c(\mathbb{Z}, C(X))$  then coincides with the algebra of polynomials in  $\delta$  with coefficients in  $C(X)$ .

The  $C^*$ -algebra  $C(X)$  can be shown to be isomorphic to the  $C^*$ -algebra  $C(X)\delta^0$  inside the normed  $*$ -algebras  $C_c(\mathbb{Z}, C(X))$  and  $l^1(\mathbb{Z}, C(X))$  having the same unit  $\delta^0$ . The mapping  $i_0 : C(X) \rightarrow C(X)\delta^0$  sending  $f \in C(X)$  to  $f\delta^0 \in C_c(\mathbb{Z}, C(X))$  is a unital  $*$ -isomorphism of the  $C^*$ -algebra  $C(X)$  onto the  $C^*$ -algebra  $C(X)\delta^0$ . We use the notation

$$(f\delta^0)[k](x) = (\delta^0 f)[k](x) = \begin{cases} f(x), & k = 0 \\ 0, & k \neq 0 \end{cases}.$$

In general, whenever it is convenient, for  $a \in l^1(\mathbb{Z}, C(X))$  and  $f \in C(X)$ , by equalities of the form  $a = f$  we will mean  $a = f\delta^0$ , and the notations  $af = a(f\delta^0)$  and  $fa = (f\delta^0)a$  will be used with products between  $a$  and  $f\delta^0$  defined by (3). The same notations will often be used for  $a$  belonging to the  $C^*$ -crossed product algebra of  $C(X)$  by  $\mathbb{Z}$  obtained as the completion of  $l^1(\mathbb{Z}, C(X))$  with respect to a certain norm. With this notation, the fundamental equality

$$(6) \quad \delta f \delta^* = \alpha(f),$$

called the *covariance relation*, holds for all  $f \in C(X)$ .

The mapping  $E : l^1(\mathbb{Z}, C(X)) \rightarrow C(X)\delta^0$  defined by  $E(b) = b[0]\delta^0$  for any element  $b \in l^1(\mathbb{Z}, C(X))$  is a projection of norm one satisfying

$$(7) \quad E(abc) = aE(b)c \text{ for all } a, c \in C(X)\delta^0, \quad (\text{module property})$$

$$(8) \quad E(b^*b) \geq 0, \quad (\text{positivity})$$

$$(9) \quad E(b^*b) = 0 \text{ implies that } b = 0 \quad (\text{faithfulness})$$

for all  $b \in l^1(\mathbb{Z}, C(X))$ . The positivity, for example, is proved as follows:

$$\begin{aligned} E(b^*b) &= (b^*b)[0]\delta^0 = \left( \sum_{k \in \mathbb{Z}} b^*[k](\cdot) \alpha^k(b[-k])(\cdot) \right) \delta^0 \\ &= \left( \sum_{k \in \mathbb{Z}} \alpha^k(\overline{b[-k]} b[-k])(\cdot) \right) \delta^0 = \left( \sum_{k \in \mathbb{Z}} \alpha^k(|b[-k]|^2(\cdot)) \right) \delta^0 \\ &= \sum_{k \in \mathbb{Z}} (|b[-k]|(\sigma^{-k}(\cdot))^2) \delta^0 \geq 0 \end{aligned}$$

where the sums converge in norm.

For any linear functional  $\varphi$  on  $C(X)$ , the mapping  $\varphi \circ i_0^{-1}$  is a linear functional on  $C(X)\delta^0$  satisfying  $(\varphi \circ i_0^{-1})(i_0(a)) = \varphi(a)$  for any  $a \in C(X)$ . Since the mapping  $a \mapsto i_0(a)$  is an isometric  $*$ -isomorphism of  $C(X)$  onto  $C(X)\delta^0$ , it follows that  $\|\varphi \circ i_0^{-1}\| = \|\varphi\|$  for any bounded  $\varphi$  on  $C(X)$ , and that  $\varphi$  is positive on  $C(X)$  if and only if  $\varphi \circ i_0^{-1}$  is positive on  $C(X)\delta^0$ .

For any positive linear functional  $\varphi$  on  $C(X)$ , the mapping  $(\varphi \circ i_0^{-1}) \circ E$  is a positive linear functional on  $l^1(\mathbb{Z}, C(X))$ . Moreover,  $\|\varphi\| = \varphi(e)$  for any positive linear functional  $\varphi$  on a Banach  $*$ -algebra with the unit  $e$ . Since

$$\begin{aligned} \|(\varphi \circ i_0^{-1}) \circ E\| &= (\varphi \circ i_0^{-1})(E(\delta^0)) = (\varphi \circ i_0^{-1})(\delta^0) = \\ &= (\varphi \circ i_0^{-1})(i_0(\mathbf{1}_{C(X)})) = \varphi(\mathbf{1}_{C(X)}) = \|\varphi\|, \end{aligned}$$

the functional  $\varphi$  is a state on  $C(X)$ , i.e. a positive linear functional with  $\|\varphi\| = 1$ , if and only if  $(\varphi \circ i_0^{-1}) \circ E$  is a state on  $l^1(\mathbb{Z}, C(X))$ .

A set of states on a Banach  $*$ -algebra  $A$  is said to contain sufficiently many states if for any non-zero  $a \in A$  there exists a state  $\varphi$  from this set such that  $\varphi(a^*a) \neq 0$ .

There are sufficiently many states on any  $C^*$ -algebra, and in particular on  $C(X)$  and on its isomorphic copy  $C(X)\delta^0$ . By faithfulness of the projection  $E$ , the set

$$\{(\varphi \circ i_0^{-1}) \circ E \mid \varphi \text{ is a state on } C(X)\}$$

of states on  $l^1(\mathbb{Z}, C(X))$  contains sufficiently many states. As a Banach  $*$ -algebra with sufficiently many states,  $l^1(\mathbb{Z}, C(X))$  has sufficiently many representations, i.e. for any non-zero  $b \in l^1(\mathbb{Z}, C(X))$  there is a representation  $\pi$  with  $\pi(b) \neq 0$ . Then it can be shown that

$$\|b\|_\infty = \sup\{\|\pi(b)\| \mid \pi \text{ is a representation of } l^1(\mathbb{Z}, C(X))\} \leq \|b\|_1$$

defines a  $C^*$ -norm on  $l^1(\mathbb{Z}, C(X))$ .

The  $C^*$ -algebra obtained as the completion of  $l^1(\mathbb{Z}, C(X))$  with respect to the norm  $\|\cdot\|_\infty$  is called the  $C^*$ -crossed product of  $C(X)$  by  $\mathbb{Z}$  with respect to the action of  $\alpha$ , or the transformation group  $C^*$ -algebra associated with the dynamical system  $\Sigma = (X, \sigma)$ . Depending on which of those two terminologies used, this algebra is denoted either by  $C(X) \rtimes_\alpha \mathbb{Z}$  or by  $A(\Sigma)$ . The  $C^*$ -algebra  $A(\Sigma)$  coincides with the closed linear span of all polynomial expressions built of  $\delta$ ,  $\delta^* = \delta^{-1}$  and also of elements from  $C(X)$ , or to be more precise from  $C(X)\delta^0$ . Because of the covariance relation (6), all  $\delta$  and  $\delta^* = \delta^{-1}$  in any such polynomial expression can be moved to the right of all elements of  $C(X)$ . Thus any polynomial expression built of  $\delta$ ,  $\delta^* = \delta^{-1}$  and of elements of  $C(X)$  is equal to a generalized polynomial in  $\delta$ , that is to an element of the form  $\sum_{j=-n}^{j=n} f_j \delta^j$ . Consequently, the  $C^*$ -algebra  $A(\Sigma)$  can be viewed as a closed linear span of generalized polynomials in  $\delta$  over  $C(X)$ . The projection  $E$  can be extended from  $l^1(\mathbb{Z}, C(X))$  to  $A(\Sigma) = C(X) \rtimes_\alpha \mathbb{Z}$  with the property of being faithful and with  $\|E\| = 1$ . For an element  $a$  of  $A(\Sigma)$ , the  $n$ 'th generalized Fourier coefficient  $a(n)$  is defined as  $E(a(\delta^*)^n)$ .

If  $\pi$  is a  $*$ -representation of the  $C^*$ -algebra  $A(\Sigma)$  on a Hilbert space  $H_\pi$ , then  $\pi' = \pi \circ i_0$  is a  $*$ -representation of the  $C^*$ -algebra  $C(X)$  on  $H_\pi$ . If one is given a  $*$ -representation  $\pi'$  of the  $C^*$ -algebra  $C(X)$  on  $H_\pi$ , then  $\pi = \pi' \circ i_0^{-1}$  is a  $*$ -representation of  $C(X)\delta^0$  on  $H_\pi$ . Moreover,  $\pi'(f) = (\pi \circ i_0)(f) = \pi(f\delta^0)$  for any  $f \in C(X)$ . With this in mind, for simplicity of notations, if  $\pi$  is a  $*$ -representation of the  $C^*$ -algebra  $A(\Sigma)$  and  $f \in C(X)$ , then by  $\pi(f)$  we will always mean  $\pi(f \circ \delta^0)$ .

If  $\pi$  is a  $*$ -representation of the  $C^*$ -algebra  $A(\Sigma)$  on a Hilbert space  $H_\pi$ , then the unitary operator  $u = \pi(\delta)$  and the commutative set (algebra) of bounded operators  $\pi(C(X))$  on  $H_\pi$  satisfy the set of commutation relations

$$(10) \quad u\pi(f)u^* = \pi(\alpha(f))$$

called covariance relations or covariance relations for a set of operators, as they are obtained by applying the  $*$ -representation  $\pi$  to both sides of the covariance relation (6) in the algebra  $A(\Sigma)$ . A pair  $(\pi, u)$  consisting of a  $*$ -representation of the  $C^*$ -algebra  $C(X)$  on a Hilbert space  $H$ , and a unitary operator  $u$  on  $H$  satisfying the covariance relations (10) is called a covariant  $*$ -representation or simply a covariant representation of the system  $(C(X), \alpha, \mathbb{Z})$ . So, any  $*$ -representation of the  $C^*$ -algebra  $A(\Sigma) = C(X) \rtimes_\alpha \mathbb{Z}$  gives rise, via restriction, to a covariant representation of the system  $(C(X), \alpha, \mathbb{Z})$ . Moreover, this covariant representation of  $(C(X), \alpha, \mathbb{Z})$  defines uniquely the  $*$ -representation of the

$C^*$ -algebra  $A(\sum)$ , and every covariant representation of the system  $(C(X), \alpha, \mathbb{Z})$  is obtained by restriction from a  $*$ -representation of the  $C^*$ -algebra  $A(\sum) = C(X) \rtimes_{\alpha} \mathbb{Z}$ . In other words, there is a one-to-one correspondence between covariant representations of the system  $(C(X), \alpha, \mathbb{Z})$ , and  $*$ -representations of the  $C^*$ -algebra  $A(\sum) = C(X) \rtimes_{\alpha} \mathbb{Z}$ . Thus the  $*$ -representations of the  $C^*$ -crossed product  $A(\sum) = C(X) \rtimes_{\alpha} \mathbb{Z}$  can be completely described and studied in terms of the covariant representation of the system  $(C(X), \alpha, \mathbb{Z})$ , that is in terms of families of operators satisfying the covariance commutation relations (10) and the corresponding involution conditions. If  $(\pi, u)$  is a covariant representation of the system  $(C(X), \alpha, \mathbb{Z})$ , then the corresponding  $*$ -representation of the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$  transforms a generalized polynomial  $\sum_{j=-n}^n f_j \delta^j$  into the operator  $\sum_{j=-n}^n \pi(f_j) u^j$ .

### 3. $C^*$ -ALGEBRAS OF NON-INVERTIBLE DYNAMICAL SYSTEMS

The  $C^*$ -crossed product by  $\mathbb{Z}$  is an important way to associate a  $C^*$ -algebra to an invertible dynamical system. There are several ways to generalize this construction to non-invertible dynamical systems. One of these is due to Exel. It relies on transfer operators.

We will here give a short description of Exel's construction:

**Definition 1.** A  $C^*$ -dynamical system is a pair  $(A, \alpha)$  of a unital  $C^*$ -algebra  $A$  and an  $*$ -endomorphism  $\alpha : A \rightarrow A$ .

**Definition 2.** A transfer operator for the  $C^*$ -dynamical system  $(A, \alpha)$  is a continuous linear map  $\mathcal{L} : A \rightarrow A$  such that

- (1)  $\mathcal{L}$  is positive in the sense that  $\mathcal{L}(A_+) \subseteq A_+$ ,
- (2)  $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$  for all  $a, b \in A$ .

**Definition 3.** Given a  $C^*$ -dynamical system  $(A, \alpha)$  and a transfer operator  $\mathcal{L}$  of  $(A, \alpha)$ , we let  $\mathcal{T}(A, \alpha, \mathcal{L})$  be the universal unital  $C^*$ -algebra generated by a copy of  $A$  and an element  $S$  subject to the relations

- (1)  $Sa = \alpha(a)S$ ,
- (2)  $S^*aS = \mathcal{L}(a)$ ,

for all  $a \in A$ .

Using [1], it is easy to see that relations are admissible and thus that  $\mathcal{T}(A, \alpha, \mathcal{L})$  exists.

It is proved in [16, Corollary 3.5] that the standard embedding of  $A$  into  $\mathcal{T}(A, \alpha, \mathcal{L})$  is injective. We will therefore from now on view  $A$  as a  $C^*$ -subalgebra of  $\mathcal{T}(A, \alpha, \mathcal{L})$ .

**Definition 4.** By a redundancy we will mean a pair  $(a, k) \in A \times \overline{ASS^*A}$  such that  $abS = kbS$  for all  $b \in A$ .

**Definition 5.** The crossed product  $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is the quotient of  $\mathcal{T}(A, \alpha, \mathcal{L})$  by the closed two-sided ideal generated by the set of differences  $a - k$ , for all redundancies  $(a, k)$  such that  $a \in \overline{A\alpha(A)A}$ .

We will denote the quotient map from  $\mathcal{T}(A, \alpha, \mathcal{L})$  to  $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  by  $\rho$ .

If  $(A, \alpha)$  is an invertible  $C^*$ -dynamical system, meaning that  $\alpha$  is an automorphism, then  $\alpha^{-1}$  is a transfer operator for  $(A, \alpha)$ .

Let us consider  $\mathcal{T}(A, \alpha, \alpha^{-1})$ . It follows from (2) that  $S^*S = I$ , where  $I$  denotes the unit of  $A$ . For all  $b \in A$ ,

$$SS^*bS = S\alpha^{-1}(b) = bS = IbS,$$

so  $(I, SS^*)$  is a redundancy. Thus  $\rho(S)$  is a unitary which satisfies

$$\rho(S)\rho(a)\rho(S)^* = \rho(\alpha(a))$$

for all  $a \in A$ . In other words,  $(\rho, \rho(S))$  is a covariant representation of  $(A, \mathbb{Z}, \alpha)$ .

On the other hand, in  $A \rtimes_{\alpha} \mathbb{Z}$ ,  $\delta_1$  satisfies

- (1)  $\delta_1 a = \alpha(a)\delta_1$ ,
- (2)  $\delta_1^* a \delta_1 = \alpha^{-1}(a)$ ,

for all  $a \in A$ , and if  $ab\delta_1 = kb\delta_1$  for all  $b \in A$ , then

$$a = aI = aI\delta_1\delta_1^* = kI\delta_1\delta_1^* = kI = k,$$

so  $a - k = 0$  for all redundancies  $(a, k)$ , and thus  $A \rtimes_{\alpha} \mathbb{Z}$  is isomorphic to  $A \rtimes_{\alpha, \alpha^{-1}} \mathbb{N}$ .

#### 4. SHIFT SPACES

For an introduction to shift spaces see [35].

Let  $\mathfrak{a}$  be a finite set endowed with the discrete topology. We will call this set the alphabet. Let  $\mathfrak{a}_0^{\mathbb{N}}$  be the infinite product spaces  $\prod_{n=0}^{\infty} \mathfrak{a}$  endowed with the product topology. The transformation  $\sigma$  on  $\mathfrak{a}_0^{\mathbb{N}}$  given by  $(\sigma(x))_i = x_{i+1}$ ,  $i \in \mathbb{N}_0$  is called the *shift*. Let  $\mathbf{X}$  be a shift invariant closed subset of  $\mathfrak{a}_0^{\mathbb{N}}$  (by shift invariant we mean that  $\sigma(\mathbf{X}) \subseteq \mathbf{X}$ , not necessarily  $\sigma(\mathbf{X}) = \mathbf{X}$ ). The topological dynamical system  $(\mathbf{X}, \sigma|_{\mathbf{X}})$  is called a *shift space*. We will denote  $\sigma|_{\mathbf{X}}$  by  $\sigma_{\mathbf{X}}$  or  $\sigma$  for simplicity, and on occasion the alphabet  $\mathfrak{a}$  by  $\mathfrak{a}_{\mathbf{X}}$ .

We denote the  $n$ -fold composition of  $\sigma$  with itself by  $\sigma^n$ , and we denote the preimage of a set  $X$  under  $\sigma^n$  by  $\sigma^{-n}(X)$ .

A finite sequence  $u = (u_1, \dots, u_k)$  of elements  $u_i \in \mathfrak{a}$  is called a *finite word*. The *length* of  $u$  is  $k$  and is denoted by  $|u|$ . We let for each  $k \in \mathbb{N}_0$ ,  $\mathfrak{a}^k$  be the set of all words with length  $k$  and we let  $\mathbf{L}^k(\mathbf{X})$  be the set of all words with length  $k$  appearing in some  $x \in \mathbf{X}$ . We set  $\mathbf{L}_l(\mathbf{X}) = \bigcup_{k=0}^l \mathbf{L}^k(\mathbf{X})$  and  $\mathbf{L}(\mathbf{X}) = \bigcup_{k=0}^{\infty} \mathbf{L}^k(\mathbf{X})$  and likewise  $\mathfrak{a}_l = \bigcup_{k=0}^l \mathfrak{a}^k$  and  $\mathfrak{a}^* = \bigcup_{k=0}^{\infty} \mathfrak{a}^k$ , where  $\mathbf{L}^0(\mathbf{X}) = \mathfrak{a}^0$  denote the set consisting of the empty word  $\epsilon$  which has length 0.  $\mathbf{L}(\mathbf{X})$  is called the *language* of  $\mathbf{X}$ . Note that  $\mathbf{L}(\mathbf{X}) \subseteq \mathfrak{a}^*$  for every shift space.

For a shift space  $\mathbf{X}$  and a word  $u \in \mathbf{L}(\mathbf{X})$  we denote by  $C_{\mathbf{X}}(u)$  the *cylinder set*

$$C_{\mathbf{X}}(u) = \{x \in \mathbf{X} \mid (x_1, x_2, \dots, x_{|u|}) = u\}.$$

It is easy to see that

$$\{C_{\mathbf{X}}(u) \mid u \in \mathbf{L}(\mathbf{X})\}$$

is a basis for the topology of  $\mathbf{X}$ , and that  $C_{\mathbf{X}}(u)$  is closed and compact for every  $u \in \mathbf{L}(\mathbf{X})$ . We will allow ourself to write  $C(u)$  instead of  $C_{\mathbf{X}}(u)$  when it is clear which shift space we are working with.

For a shift space  $\mathbf{X}$  and words  $u, v \in \mathbf{L}(\mathbf{X})$  we denote by  $C(u, v)$  the set

$$C(v) \cap \sigma^{-|v|}(\sigma^{|u|}(C(u))) = \{vx \in \mathbf{X} \mid ux \in \mathbf{X}\}.$$

What we have defined above is a *one-sided* shift space. A *two-sided* shift space is defined in the same way, except that we replace  $\mathbb{N}_0$  with  $\mathbb{Z}$ : Let  $\mathfrak{a}^{\mathbb{Z}}$  be the infinite product spaces  $\prod_{n=-\infty}^{\infty} \mathfrak{a}$  endowed with the product topology, and let  $\sigma$  be the transformation on



$\mathfrak{a}^{\mathbb{Z}}$  given by  $(\sigma(x))_i = x_{i+1}$ ,  $i \in \mathbb{Z}$ . A shift invariant closed subset  $\Lambda$  of  $\mathfrak{a}^{\mathbb{Z}}$  (here, by shift invariant we mean  $\sigma(\Lambda) = \Lambda$ ) is called a *two-sided shift space*. The set

$$\mathbf{X}_\Lambda = \{(x_i)_{i \in \mathbb{N}_0} \mid (x_i)_{i \in \mathbb{Z}} \in \Lambda\}$$

is a one-sided shift space, and it is called the one-sided shift space of  $\Lambda$ .

If  $\mathbf{X}$  and  $\mathbf{Y}$  are two shift spaces and  $\phi : \mathbf{X} \rightarrow \mathbf{Y}$  is a homeomorphism such that  $\phi \circ \sigma_{\mathbf{X}} = \sigma_{\mathbf{Y}} \circ \phi$ , then we say that  $\phi$  is a *conjugacy* and that  $\mathbf{X}$  and  $\mathbf{Y}$  are *conjugate* or *one-sided conjugate* if we want to emphasize that we are dealing with one-sided shift spaces. Likewise we say that two two-sided shift spaces  $\Lambda$  and  $\Gamma$  are *two-sided conjugate* if there exists a homeomorphism  $\phi : \Lambda \rightarrow \Gamma$  such that  $\phi \circ \sigma_\Lambda = \sigma_\Gamma \circ \phi$ . It is an easy exercise to prove that if  $\mathbf{X}_\Lambda$  and  $\mathbf{X}_\Gamma$  are one-sided conjugate, then  $\Lambda$  and  $\Gamma$  are two-sided conjugate.

The weaker notion of *flow equivalence* among two-sided shift spaces is also of importance here. This notion is defined using the *suspension flow space* of  $(\Lambda, \sigma)$  defined as  $S\Lambda = (\Lambda \times \mathbb{R}) / \sim$  where the equivalence relation  $\sim$  is generated by requiring that  $(x, t+1) \sim (\sigma(x), t)$ . Equipped with the quotient topology, we get a compact space with a *continuous flow* consisting of a family of maps  $(\phi_t)$  defined by  $\phi_t([x, s]) = [x, s+t]$ . We say that two two-sided shift spaces  $\Lambda$  and  $\Gamma$  are *flow equivalent* and write  $\Lambda \cong_f \Gamma$  if a homeomorphism  $F : S\Lambda \rightarrow S\Gamma$  exists with the property that for every  $x \in S\Lambda$  there is a monotonically increasing map  $f_x : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$F(\phi_t(x)) = \phi'_{f_x(t)}(F(x)).$$

In words,  $F$  takes flow orbits to flow orbits in an orientation-preserving way. It is not hard to see that two-sided conjugacy implies flow equivalence.

## 5. THE C\*-ALGEBRA ASSOCIATED WITH A SHIFT SPACE

Let  $(\mathbf{X}, \sigma)$  be a one-sided shift space. We want to define on  $C(\mathbf{X})$  a transfer operator

$$\mathcal{L}(f)(x) = \begin{cases} \frac{1}{\#\sigma^{-1}(\{x\})} \sum_{y \in \sigma^{-1}\{x\}} f(y) & \text{if } x \in \sigma(\mathbf{X}), \\ 0 & \text{if } x \notin \sigma(\mathbf{X}), \end{cases}$$

where the symbol  $\#$  is used for the cardinality of a set. But such operator might take us out of the class of continuous functions on  $\mathbf{X}$ . So we let  $\mathcal{D}_{\mathbf{X}}$  be the smallest  $C^*$ -subalgebra of the  $C^*$ -algebra of bounded functions on  $\mathbf{X}$ , containing  $C(\mathbf{X})$  and closed under  $\mathcal{L}$  and  $\alpha$ , where  $\alpha$  is the map  $f \mapsto f \circ \sigma$ .

**Lemma 6.** *The function*

$$x \mapsto \sigma^{-n}\{x\}, \quad x \in \mathbf{X}$$

*belongs to  $\mathcal{D}_{\mathbf{X}}$  for every  $n \in \mathbb{N}$ .*

*Proof.* Let  $n \in \mathbb{N}$ . The function

$$f = 1 - \mathcal{L}^n(1) + \sum_{u \in \mathfrak{a}^n} (\mathcal{L}^n(1_{C(u)}))^2$$

belongs to  $\mathcal{D}_{\mathbf{X}}$ , and since

$$f(x) = \begin{cases} \frac{1}{\#\sigma^{-n}\{x\}} & \text{if } x \in \sigma^n(\mathbf{X}), \\ 1 & \text{if } x \notin \sigma^n(\mathbf{X}), \end{cases}$$

for every  $x \in X$ ,  $f$  is invertible and  $f^{-1} \in \mathcal{D}_X$ , and so does  $f^{-1} + \mathcal{L}^n(1) - 1$ . Since

$$(f^{-1} + \mathcal{L}^n(1) - 1)(x) = \begin{cases} \#\sigma^{-n}\{x\} & \text{if } x \in \sigma^n(X), \\ 0 & \text{if } x \notin \sigma^n(X), \end{cases} = \#\sigma^{-n}\{x\}$$

for every  $x \in X$ , we are done.  $\square$

**Lemma 7.**  $\mathcal{D}_X$  is the  $C^*$ -algebra generated by  $\{1_{C(u,v)}\}_{u,v \in \mathfrak{a}^*}$ .

*Proof.* Let  $f(x) = \#\sigma^{-|u|}\{x\}$ . It then follows from Lemma 6 that  $f \in \mathcal{D}_X$ . Thus

$$1_{C(u,v)} = 1_{C(v)}\alpha^{|v|}(f\mathcal{L}^{|u|}(1_{C(u)}))$$

belongs to  $\mathcal{D}_X$  for every  $u, v \in \mathfrak{a}^*$ , and  $C^*(1_{C(u,v)} \mid u, v \in \mathfrak{a}^*) \subseteq \mathcal{D}_X$ .

In the other direction we have that since  $\{C(v)\}_{v \in \mathfrak{a}^*}$  is a basis of the topology of  $X$  consisting of clopen sets,  $\{1_{C(v)}\}_{v \in \mathfrak{a}^*}$  generates  $C(X)$  and since  $1_{C(v)} = 1_{C(\epsilon, v)}$ ,  $C(X)$  is contained in  $C^*(1_{C(u,v)} \mid u, v \in \mathfrak{a}^*)$ . Since

$$\alpha(1_{C(u,v)}) = \sum_{a \in \mathfrak{a}} 1_{C(u,av)},$$

and

$$\mathcal{L}(1_{C(u,v)}) = \left( \sum_{a \in \mathfrak{a}} (\mathcal{L}(1_{C(a)}))^2 \right) 1_{C(v_1, \epsilon)} 1_{C(u, v_2 v_3 \dots v_n)},$$

if  $v \neq \epsilon$ , and

$$\mathcal{L}(1_{C(u, \epsilon)}) = \left( \sum_{a \in \mathfrak{a}} (\mathcal{L}(1_{C(a)}))^2 \right) \left( \sum_{a \in \mathfrak{a}} 1_{C(au, \epsilon)} \right),$$

the  $C^*$ -algebra generated by  $1_{C(u,v)}$ ,  $u, v \in \mathfrak{a}^*$  is closed under  $\mathcal{L}$  and  $\alpha$  and thus contain  $\mathcal{D}_X$ .  $\square$

**Theorem 8.** The  $C^*$ -algebra  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is the universal  $C^*$ -algebra generated by partial isometries  $\{S_u\}_{u \in \mathfrak{a}^*}$  satisfying:

- (1)  $S_u S_v = S_{uv}$  for all  $u, v \in \mathfrak{a}^*$ ,
- (2) the map

$$1_{C(u,v)} \mapsto S_v S_u^* S_u S_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to a  $*$ -homomorphism from  $\mathcal{D}_X$  to  $C^*(S_u \mid u \in \mathfrak{a}^*)$ .

*Proof.* We will first show that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is generated by partial isometries  $\{S_u\}_{u \in \mathfrak{a}^*}$  satisfying (1) and (2), and then that if  $A$  is a  $C^*$ -algebra generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfying (1) and (2), then there is a  $*$ -homomorphism from  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $A$  sending  $S_u$  to  $s_u$  for all  $u \in \mathfrak{a}^*$ .

Working within  $\mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L})$  let for each  $a \in \mathfrak{a}$ ,

$$T_a = 1_{C(a)} (\alpha(f))^{1/2} S,$$

where  $f$  is the function  $x \mapsto \#\sigma^{-1}\{x\}$ , which belongs to  $\mathcal{D}_X$  by Lemma 6, and we let for each  $u = u_1 u_2 \dots u_n \in \mathfrak{a}^*$ ,

$$S_u = \rho(T_{u_1}) \rho(T_{u_2}) \dots \rho(T_{u_n}).$$

Then clearly  $\{S_u\}_{u \in \mathfrak{a}^*}$  satisfy (1).

Let  $a \in \mathfrak{a}$ ,  $g \in \mathcal{D}_X$  and  $x \in X$ . Then

$$\begin{aligned} (T_a^* g T_a)(x) &= (S^* \alpha(f) 1_{C(a)} g S)(x) \\ &= (\mathcal{L}(\alpha(f) 1_{C(a)} g))(x) \\ &= (f \mathcal{L}(1_{C(a)} g))(x) \\ &= \begin{cases} g(ax) & \text{if } ax \in X, \\ 0 & \text{if } ax \notin X. \end{cases} \end{aligned}$$

Now let  $a \in \mathfrak{a}$  and  $g, h \in \mathcal{D}_X$ . Then

$$\begin{aligned} T_a g T_a^* h S &= 1_{C(a)} (\alpha(f))^{1/2} S g S^* (\alpha(f))^{1/2} 1_{C(a)} h S \\ &= 1_{C(a)} (\alpha(f))^{1/2} S g \mathcal{L}((\alpha(f))^{1/2} 1_{C(a)} h) \\ &= 1_{C(a)} \alpha(f^{1/2} g \mathcal{L}((\alpha(f))^{1/2} 1_{C(a)} h)) S, \end{aligned}$$

and for every  $x \in X$  is

$$\left( 1_{C(a)} \alpha(f^{1/2} g \mathcal{L}((\alpha(f))^{1/2} 1_{C(a)} h)) \right)(x) = \begin{cases} g(\sigma(x)) h(x) & \text{if } x \in C(a), \\ 0 & \text{if } x \notin C(a). \end{cases}$$

Since  $S g S^* = \alpha(g) S S^*$ ,

$$T_a g T_a^* = 1_{C(a)} (\alpha(f))^{1/2} S g S^* (\alpha(f))^{1/2} 1_{C(a)} \in \overline{A S S^* A},$$

so  $(\alpha(g) 1_{C(a)}, T_a g T_a^*)$  is a redundancy, and since  $\alpha(g) 1_{C(a)} \in \overline{A \alpha(A) A}$ , it follows that  $S_a \rho(g) S_a^*$  and  $\rho(\alpha(g) 1_{C(a)})$  are equal in  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ .

Thus  $S_a^* \rho(g) S_a = \lambda_a(g)$  and  $S_a \rho(g) S_a^* = \rho(\alpha(g) 1_{C(a)})$  for every  $a \in \mathfrak{a}$  and  $g \in \mathcal{D}_X$ , where  $\lambda_a(g)$  is the map given by

$$\lambda_a(g)(x) = \begin{cases} g(ax) & \text{if } ax \in X, \\ 0 & \text{if } ax \notin X, \end{cases}$$

for  $x \in X$ , which shows that

$$\rho(1_{C(u,v)}) = S_v S_u^* S_u S_v^*$$

for every  $u, v \in \mathfrak{a}^*$ . Hence  $\{S_u\}_{u \in \mathfrak{a}^*}$  satisfy (2). To see that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is generated by  $\{S_u\}_{u \in \mathfrak{a}^*}$ , we first notice that  $\mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L})$  is generated by  $\mathcal{D}_X$  and  $S$ , and that  $\mathcal{D}_X$ , by Lemma 7, is generated by  $\{1_{C(u,v)}\}_{u,v \in \mathfrak{a}^*}$ , and then that the function  $\alpha(f)$ , where  $f$  as before is the function

$$x \mapsto \sigma^{-n}\{x\}, \quad x \in X,$$

is invertible and that  $S = \sum_{a \in \mathfrak{a}} \alpha(f)^{-1/2} T_a$ . Thus it follows that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is generated by  $\{S_u\}_{u \in \mathfrak{a}^*}$ .

Assume now that  $A$  is a  $C^*$ -algebra generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$  which satisfy (1) and (2). We then let  $s = \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a$ , where  $\phi$  is the  $*$ -homomorphism from  $\mathcal{D}_X$  to  $C^*(s_u \mid u \in \mathfrak{a}^*)$ , which extends the map

$$1_{C(u,v)} \mapsto s_v s_u^* s_u s_v^*, \quad u, v \in \mathfrak{a}^*.$$

Observe first that if  $a, b \in \mathfrak{a}$  and  $a \neq b$ , then  $s_a^* s_b = 0$ , because

$$s_a^* s_b = s_a^* s_a s_a^* s_b s_b^* s_b = s_a^* \phi(1_{C(a)} 1_{C(b)}) s_b.$$

Let then  $a \in \mathfrak{a}$  and  $u, v \in \mathfrak{a}^*$ . We then have that if  $v \neq \epsilon$ , then

$$\begin{aligned}
s_a^* \phi(1_{C(u,v)}) s_a &= s_a^* s_v s_u^* s_u s_v^* s_a \\
&= s_a^* s_{v_1} s_{v_2 v_3 \dots v_{|v|}} s_u^* s_u s_{v_2 v_3 \dots v_{|v|}}^* s_{v_1}^* s_a \\
&= \begin{cases} \phi(1_{C(v_1, \epsilon)} 1_{C(u, v_2 v_3 \dots v_{|v|})}) & \text{if } a = v_1, \\ 0 & \text{if } a \neq v_1 \end{cases} \\
&= \phi(\lambda_a(1_{C(u,v)})),
\end{aligned}$$

and if  $v = \epsilon$ , then

$$\begin{aligned}
s_a^* \phi(1_{C(u,v)}) s_a &= s_a^* s_u^* s_u s_a \\
&= s_{ua}^* s_{ua} \\
&= \phi(1_{C(ua, \epsilon)}) \\
&= \phi(\lambda_a(1_{C(u,v)})).
\end{aligned}$$

Since  $\mathcal{D}_X$  is generated by  $1_{C(u,v)}$ ,  $u, v \in \mathfrak{a}^*$ , this shows that  $s_a^* \phi(g) s_a = \phi(\lambda_a(g))$  for each  $a \in \mathfrak{a}$  and every  $g \in \mathcal{D}_X$ .

Let  $u, v \in \mathfrak{a}^*$ . Then

$$\begin{aligned}
\phi(\alpha(1_{C(u,v)})) s &= \sum_{a \in \mathfrak{a}} \phi(1_{C(u,av)}) s \\
&= \sum_{a \in \mathfrak{a}} \phi(1_{C(u,av)}) \sum_{b \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_b \\
&= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)}) s_b s_b^* s_b \\
&= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)} 1_{C(b)}) s_b \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)}) s_a \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_{av} s_u^* s_u s_{av}^* s_a \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a s_v s_u^* s_u s_v^* s_a^* s_a \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a \phi(1_{C(u,v)} 1_{C(a, \epsilon)}) \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a \phi(1_{C(a, \epsilon)} 1_{C(u,v)}) \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a s_a^* s_a \phi(1_{C(u,v)}) \\
&= \sum_{a \in \mathfrak{a}} \phi(\alpha(f)^{-1/2}) s_a \phi(1_{C(u,v)}) \\
&= s \phi(1_{C(u,v)}),
\end{aligned}$$

and

$$\begin{aligned}
s^* \phi(1_{C(u,v)}) s &= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} s_a^* \phi(\alpha(f)^{-1/2} 1_{C(u,v)} \alpha(f)^{-1/2}) s_b \\
&= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} s_a^* \phi(\alpha(f)^{-1} 1_{C(a)} 1_{C(b)} 1_{C(u,v)}) s_b \\
&= \phi(\alpha(f)^{-1}) \sum_{a \in \mathfrak{a}} s_a^* \phi(1_{C(u,v)}) s_a^* \\
&= \phi(\alpha(f)^{-1}) \sum_{a \in \mathfrak{a}} s_a^* s_v s_u^* s_u s_v^* s_a^* \\
&= \phi(\alpha(f)^{-1}) \sum_{a \in \mathfrak{a}} s_a^* s_{v_1} s_{v_2 v_3 \dots v_{|v|}} s_u^* s_u s_{v_2 v_3 \dots v_{|v|}} s_{v_1}^* s_a^* \\
&= \phi(\alpha(f)^{-1}) s_{v_1}^* s_{v_1} s_{v_2 v_3 \dots v_{|v|}} s_u^* s_u s_{v_2 v_3 \dots v_{|v|}} s_{v_1}^* s_{v_1}^* \\
&= \phi(\alpha(f)^{-1} 1_{C(v_1, \epsilon)} 1_{C(v_2 v_3 \dots v_{|v|}, u)}) \\
&= \phi(\mathcal{L}(1_{C(v, u)}))
\end{aligned}$$

and since  $\mathcal{D}_X$  is generated by  $1_{C(u,v)}$ ,  $u, v \in \mathfrak{a}^*$ , this shows that  $\phi(\alpha(g))s = s\phi(g)$  for every  $g \in \mathcal{D}_X$ .

Thus it follows from the universal property of  $\mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L})$ , that there exists a  $*$ -homomorphism  $\psi$  from  $\mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L})$  to  $A$  which maps  $g$  to  $\phi(g)$  for  $g \in \mathcal{D}_X$  and  $S$  to  $s$ . We will now show that  $\psi$  vanishes on the closed two-sided ideal generated by the set of differences  $g - k$ , for all redundancies  $(g, k)$  such that  $g \in \overline{\mathcal{D}_X \alpha(\mathcal{D}_X) \mathcal{D}_X}$ , and thus that it factors through the quotient and yields a  $*$ -homomorphism  $\tilde{\psi} : \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \rightarrow A$  such that  $\tilde{\psi}(\rho(g)) = \phi(g)$  and  $\tilde{\psi}(\rho(S)) = s$ , and hence

$$\begin{aligned}
\tilde{\psi}(S_a) &= \tilde{\psi}(\rho(T_a)) \\
&= \tilde{\psi}(\rho(1_{C(a)}(\alpha(f)^{1/2})S)) \\
&= \phi(1_{C(a)}(\alpha(f))^{1/2})s \\
&= \phi(1_{C(a)}(\alpha(f))^{1/2}) \sum_{b \in \mathfrak{a}} \phi((\alpha(f))^{-1/2}) s_b \\
&= \sum_{b \in \mathfrak{a}} \phi(1_{C(a)} 1_{C(b)}) s_b \\
&= \phi(1_{C(a)}) s_a \\
&= s_a s_a^* s_a \\
&= s_a
\end{aligned}$$

for all  $a \in \mathfrak{a}$ , and thus  $\tilde{\psi}(S_u) = s_u$  for every  $u \in \mathfrak{a}^*$ .

Assume that  $g \in \overline{\mathcal{D}_X \alpha(\mathcal{D}_X) \mathcal{D}_X}$ , that  $k \in \overline{\mathcal{D}_X S S^* \mathcal{D}_X}$  and  $ghS = khS$  for every  $h \in \mathcal{D}_X$ . Then

$$\begin{aligned}
\psi(g) &= \psi\left(g \sum_{a \in \mathfrak{a}} 1_{C(a)}\right) = \psi(g) \sum_{a \in \mathfrak{a}} s_a s_a^* \\
&= \psi(g) \sum_{a \in \mathfrak{a}} \phi(1_{C(a)}(\alpha(f))^{1/2}) s_a s_a^* \\
&= \sum_{a \in \mathfrak{a}} \psi(g 1_{C(a)}(\alpha(f))^{1/2} S) s_a s_a^* \\
&= \sum_{a \in \mathfrak{a}} \psi(k 1_{C(a)}(\alpha(f))^{1/2} S) s_a s_a^* \\
&= \psi(k) \sum_{a \in \mathfrak{a}} \phi(1_{C(a)}(\alpha(f))^{1/2}) s_a s_a^* \\
&= \psi(k) \sum_{a \in \mathfrak{a}} s_a s_a^* = \psi\left(k \sum_{a \in \mathfrak{a}} 1_{C(a)}\right) \\
&= \psi(k),
\end{aligned}$$

so  $\psi$  vanishes on the closed two-sided ideal generated by the set of differences  $g - k$ , for all redundancies  $(g, k)$  such that  $g \in \overline{\mathcal{D}_X \alpha(\mathcal{D}_X) \mathcal{D}_X}$ .  $\square$

## 6. A REPRESENTATION OF $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$

Let  $\mathbf{X}$  be a shift space, let  $\mathbf{H}_X$  be the Hilbert space  $l_2(\mathbf{X})$  and let  $\{e_x\}_{x \in \mathbf{X}}$  be an orthonormal basis for  $\mathbf{H}_X$ . Let for every  $u \in \mathfrak{a}^*$ ,  $s_u$  be the operator on  $\mathbf{H}_X$  defined by

$$s_u(e_x) = \begin{cases} e_{ux} & \text{if } ux \in \mathbf{X}, \\ 0 & \text{if } ux \notin \mathbf{X}. \end{cases}$$

We leave it to the reader to check that the operators  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfy condition (1) and (2) of Theorem 8. Thus there exists a  $*$ -homomorphism  $\phi : \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \rightarrow C^*(s_u \mid u \in \mathfrak{a}^*)$  such that  $\phi(S_u) = s_u$  for every  $u \in \mathfrak{a}^*$ . In other words,  $S_u \mapsto s_u$  is a representation of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  on the Hilbert space  $\mathbf{H}_X$ .

This representation is in general not faithful. If for example  $\mathbf{X}$  only consist of one word, then  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is isomorphic to  $C(\mathbb{T})$ , whereas  $C^*(s_u \mid u \in \mathfrak{a}^*)$  is isomorphic to  $\mathbb{C}$ . We will in section 9 see that if the shift space  $\mathbf{X}$  satisfies a certain condition (I), then the representation  $\phi$  is injective. We will in section 9 construct a representation of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  which is faithful for every shift space  $\mathbf{X}$ .

Although the  $*$ -homomorphism  $\phi : \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \rightarrow C^*(s_u \mid u \in \mathfrak{a}^*)$  is not in general injective the restriction of  $\phi$  to  $\mathcal{D}_X$  is, and so it follows from the universal property of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , that also the restriction of  $\rho : \mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L}) \rightarrow \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $\mathcal{D}_X$  is injective. Thus we will allow ourselves to view  $\mathcal{D}_X$  as a sub-algebra of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ . We then have

$$1_{C(u,v)} = S_v S_u^* S_u S_v^*$$

for all  $u, v \in \mathfrak{a}^*$ .

## 7. $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ 'S RELATIONSHIP WITH OTHER $C^*$ -ALGEBRAS ASSOCIATED TO SHIFT SPACES

As mentioned in the introduction, other  $C^*$ -algebras have been associated to shift spaces. We will in this section look at the relation between these  $C^*$ -algebras and  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ .

As far as the authors know, three different construction of  $C^*$ -algebras associated to shift spaces appears in the literature. These are:

- The  $C^*$ -algebra  $\mathcal{O}_\Lambda$  defined in [23],
- the  $C^*$ -algebra  $\mathcal{O}_\Lambda$  defined in [13],
- the  $C^*$ -algebra  $\mathcal{O}_X$  defined in [5].

These are all  $C^*$ -algebras generated by partial isometries  $\{S_a\}_{a \in \mathfrak{a}}$ , where  $\mathfrak{a}$  is the alphabet of the shift space in question. The two first  $C^*$ -algebras are defined for every two-sided shift space  $\Lambda$ , whereas the last one is defined for every one-sided shift space  $X$ .

We will in this section see, that there for every one-sided shift space  $X$  exists a  $*$ -isomorphism between  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and the  $C^*$ -algebra  $\mathcal{O}_X$  defined in [5] which maps  $S_a$  to  $S_a$  for every  $a \in \mathfrak{a}$ , and that there for every two-sided shift space  $\Lambda$  exist a surjective  $*$ -homomorphism from the  $C^*$ -algebra  $\mathcal{O}_\Lambda$  defined in [23] to  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  which maps  $S_a$  to  $S_a$  for every  $a \in \mathfrak{a}$ , and a surjective  $*$ -homomorphism from  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to the  $C^*$ -algebra  $\mathcal{O}_\Lambda$  defined in [13] which maps  $S_a$  to  $S_a$  for every  $a \in \mathfrak{a}$ . The first of these surjective  $*$ -homomorphisms is injective if  $\Lambda$  satisfy the condition (\*) defined in [13], and the second surjective  $*$ -homomorphism is injective if  $\Lambda$  satisfy the condition (I) in Section 9.

**Remark 9.** In [5], a  $C^*$ -algebra  $\mathcal{O}_X$  has been constructed by using  $C^*$ -correspondences and Cuntz-Pimsner algebras for every shift space  $X$ . It follows from Theorem 8 and [5, Remark 7.4] that  $\mathcal{O}_X$  is isomorphic to  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  for every one-sided shift space  $X$ . Thus it follows from [5, Remark 7.4] that for every two-sided shift space  $\Lambda$ , the algebra  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  satisfy all of the results the algebra  $\mathcal{O}_\Lambda$  is claimed to satisfy in [23–25, 28–33].

**Remark 10.** In [13] a  $C^*$ -algebra  $\mathcal{O}_\Lambda$  has been defined for every two-sided shift space by defining operators on the Hilbert space  $l_2(X_\Lambda)$ . These operators are identical to the operators  $s_u$  defined in section 6 for  $X$  equal to the one-sided shift space  $X_\Lambda$  associated to  $\Lambda$ . Thus we have for every two-sided shift space  $\Lambda$  a surjective  $*$ -homomorphism from  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $\mathcal{O}_\Lambda$  which is injective if  $\Lambda$  satisfies condition (I), and we also know that there are examples of two-sided shift spaces (for instance the shift only consisting of one point) for which the  $*$ -homomorphism is not injective.

As we have mentioned before, our  $C^*$ -algebra  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  satisfies all of the results that the algebra  $\mathcal{O}_\Lambda$  is claimed to satisfy [23–25, 28–33], whereas the  $C^*$ -algebra  $\mathcal{O}_\Lambda$  originally defined in [30], does not. The latter  $C^*$ -algebra have been properly characterized in [13] (where it is called  $\mathcal{O}_\Lambda^*$ ). We will now use this characterization to show that there for every two-sided shift space  $\Lambda$  exists a surjective  $*$ -homomorphism from  $\mathcal{O}_\Lambda$  to  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ .

Let for every  $l \in \mathbb{N}_0$ ,  $\mathcal{A}_l^*$  be the  $C^*$ -subalgebra of  $\mathcal{O}_\Lambda$  generated by  $\{S_u^* S_u\}_{u \in \mathfrak{a}_l}$ , and let  $\mathcal{A}_\Lambda^*$  be the  $C^*$ -subalgebra of  $\mathcal{O}_\Lambda$  generated by  $\{S_u^* S_u\}_{u \in \mathfrak{a}}$ . Notice that

$$\mathcal{A}_\Lambda^* = \overline{\bigcup_{l \in \mathbb{N}_0} \mathcal{A}_l^*}.$$

The key to characterizing  $\mathcal{O}_\Lambda$  is to describe  $\mathcal{A}_l^*$  and  $\mathcal{A}_\Lambda^*$ . This is done in this way:

Let for every  $l \in \mathbb{N}_0$  and every  $u \in \mathbf{L}(\Lambda)$ ,

$$\mathcal{P}_l(u) = \{v \in \mathfrak{a}_l \mid vu \in \mathbf{L}(\Lambda)\}.$$

We then define an equivalence relation  $\sim_l$  on  $\mathbf{L}(\Lambda)$  called *l-past equivalence* in this way:

$$u \sim_l v \iff \mathcal{P}_l(u) = \mathcal{P}_l(v).$$

We denote the  $l$ -past equivalence class containing  $u$  by  $[u]_l$ , and we let

$$\mathbf{L}_l^*(\Lambda) = \{u \in \mathfrak{a}_l^* \mid \text{the cardinality of } [u]_l \text{ is infinite}\},$$

and  $\Omega_l^* = \mathbf{L}_l^*/\sim_l$ . Since  $\mathfrak{a}_l^*$  is finite, so is  $\Omega_l^*$ . We equip  $\Omega_l^*$  with the discrete topology (so  $C(\Omega_l^*) \cong \mathbb{C}^{m^*(l)}$ , where  $m^*(l)$  is the number of elements of  $l$ -past equivalence classes).

We then have:

**Lemma 11** (cf. [13, Lemma 2.9]). *The map*

$$1_{\{[u]_l\}} \mapsto 1_{[u]_l}, \quad u \in \mathbf{L}_l^*(\Lambda)$$

*extends to a \*-isomorphism between  $C(\Omega_l^*)$  and  $\mathcal{A}_l^*$ .*

We will now make the corresponding characterization of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ :

Let  $X$  be a one-sided shift space. We let for every  $l \in \mathbb{N}_0$ ,  $\mathcal{A}_l$  be the  $C^*$ -subalgebra of  $\mathcal{D}_X$  generated by  $\{1_{C(v, \epsilon)}\}_{v \in \mathfrak{a}_l^*}$ , and we let  $\mathcal{A}_X$  be the  $C^*$ -subalgebra of  $\mathcal{D}_X$  generated by  $\{1_{C(v, \epsilon)}\}_{v \in \mathfrak{a}^*}$ . Notice that

$$\mathcal{A}_X = \overline{\bigcup_{l \in \mathbb{N}_0} \mathcal{A}_l}.$$

Following Matsumoto (cf. [25]), we let for every  $l \in \mathbb{N}$  and every  $x \in X$ ,

$$\mathcal{P}_l(x) = \{u \in \mathfrak{a}_l^* \mid ux \in X\}.$$

We then define an equivalence relation  $\sim_l$  on  $X$  called *l-past equivalence* in this way:

$$x \sim_l y \iff \mathcal{P}_l(x) = \mathcal{P}_l(y).$$

We let  $\Omega_l = X/\sim_l$ , and denote the  $l$ -past equivalence class containing  $x$  by  $[x]_l$ . Since  $\mathfrak{a}_l^*$  is finite, so is  $\Omega_l$ . We equip  $\Omega_l$  with the discrete topology (so  $C(\Omega_l) \cong \mathbb{C}^{m(l)}$ , where  $m(l)$  is the number of elements of  $l$ -past equivalence classes). Since

$$[x]_l = \left( \bigcap_{u \in \mathcal{P}_l(x)} C(u, \epsilon) \right) \cap \left( \bigcap_{v \in \mathfrak{a}_l^* \setminus \mathcal{P}_l(x)} X \setminus C(v, \epsilon) \right),$$

the function  $1_{[x]_l}$  belongs to  $\mathcal{A}_l$ , and  $\{1_{[x]_l}\}_{x \in X}$  generates  $\mathcal{A}_l$ . Thus

$$1_{\{[x]_l\}} \mapsto 1_{[x]_l}$$

is a \*-isomorphism between  $C(\Omega_l)$  and  $\mathcal{A}_l$ , which extends to an isomorphism between  $C(\Omega_X)$  and  $\mathcal{A}_X$ .

Consider the condition:

(\*) : There exists for each  $l \in \mathbb{N}_0$  and each infinite sequence of admissible words  $\{u_i\}_{i \in \mathbb{N}}$  satisfying  $\mathcal{P}_l(u_i) = \mathcal{P}_l(u_j)$  for all  $i, j \in \mathbb{N}$ , an  $x \in X_\Lambda$  such that

$$\mathcal{P}_l(x) = \mathcal{P}_l(u_i)$$

for all  $i \in \mathbb{N}$ .

It follows from [13, Corollary 3.3] that there is a surjective \*-homomorphism from  $\mathcal{A}_\Lambda^*$  to  $\mathcal{A}_{X_\Lambda}$ , and that this \*-homomorphism is injective if and only if  $\Lambda$  satisfies the condition (\*).



As a consequence of this, we get that for every two-sided shift space  $\Lambda$  exists a surjective  $*$ -homomorphism from  $\mathcal{O}_\Lambda$  to  $\mathcal{D}_{X_\Lambda \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$ , and that this  $*$ -homomorphism is injective if  $\Lambda$  satisfies the condition (\*).

There is in [13] an example of a sofic shift space  $\Lambda$  for which  $\mathcal{O}_\Lambda$  and  $\mathcal{D}_{X_\Lambda \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$  are not isomorphic.

## 8. GENERALIZATION OF THE CUNTZ-KRIEGER ALGEBRAS

We are now able to show that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  in fact is a generalization of the Cuntz-Krieger algebras. Actual we will prove that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is a generalization of the universal Cuntz-Krieger algebra  $\mathcal{AO}_A$  that An Huef and Raeburn have constructed in [18].

**Theorem 12.** *Let  $A$  be a  $n \times n$ -matrix with entries in  $\{0, 1\}$  and no zero rows, and let  $X_A$  be the one-sided shift spaces*

$$\{(x_i)_{i \in \mathbb{N}_0} \in \{1, 2, \dots, n\}_{i=0}^\infty \mid \forall i \in \mathbb{N}_0 : A(x_i, x_{i+1}) = 1\}.$$

*Then  $\mathcal{D}_{X_A \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$  is generated by partial isometries  $\{S_i\}_{i \in \{1, 2, \dots, n\}}$  that satisfy*

$$\sum_{j=1}^n S_j S_j^* = I,$$

*and*

$$S_i^* S_i = \sum_{j=1}^n A(i, j) S_j S_j^*$$

*for every  $i \in \{1, 2, \dots, n\}$ .*

*If  $X$  is a unital  $C^*$ -algebra such that there exists a set of partial isometries  $\{T_i\}_{i \in \{1, 2, \dots, n\}}$  in  $X$  that satisfy*

$$\sum_{j=1}^n T_j T_j^* = I,$$

*and*

$$T_i^* T_i = \sum_{j=1}^n A(i, j) T_j T_j^*$$

*for every  $i \in \{1, 2, \dots, n\}$ ; then there exists a  $*$ -homomorphism form  $\mathcal{D}_{X_A \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$  to  $X$  sending  $S_i$  to  $T_i$  for every  $i \in \{1, 2, \dots, n\}$ .*

*Proof.* Since  $X_A$  is the disjoint union of  $C(j)$ ,  $j \in \{1, 2, \dots, n\}$ ,

$$\sum_{j=1}^n S_j S_j^* = I,$$

and since for every  $i \in \{1, 2, \dots, n\}$ ,  $C(i, \epsilon)$  is the disjoint union of those  $C(j)$ 's, where  $A(i, j) = 1$ ,

$$S_i^* S_i = 1_{C(i, \epsilon)} = \sum_{j=1}^n A(i, j) 1_{C(j)} = \sum_{j=1}^n A(i, j) S_j S_j^*.$$

The  $C^*$ -algebra  $\mathcal{D}_{X_A \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$  is generated by partial isometries  $\{S_u\}_{u \in \{1, 2, \dots, n\}^*}$ , but since these partial isometries satisfy  $S_u S_v = S_{uv}$  for all  $u, v \in \{1, 2, \dots, n\}^*$ ,  $\{S_i\}_{i \in \{1, 2, \dots, n\}}$  generates the whole  $\mathcal{D}_{X_A \rtimes_{\alpha, \mathcal{L}}} \mathbb{N}$ .

Let  $X$  be a unital  $C^*$ -algebra such that there exist partial isometries  $T_i$ ,  $i \in \{1, 2, \dots, n\}$  in  $X$  that satisfy

$$\sum_{j=1}^n T_j T_j^* = I,$$

and

$$T_i^* T_i = \sum_{j=1}^n A(i, j) T_j T_j^*$$

for every  $i \in \{1, 2, \dots, n\}$ . We let  $T_\epsilon = I$  and we let for every  $u = u_1 u_2 \cdots u_n \in \{1, 2, \dots, n\}^* \setminus \{\epsilon\}$ ,  $T_u$  be  $T_u = T_{u_1} T_{u_2} \cdots T_{u_n}$ , and we will then show that

- (1)  $T_u T_v = T_{uv}$  for all  $u, v \in \{1, 2, \dots, n\}^*$ ,
- (2) the map

$$1_{C(u, v)} \mapsto T_v T_u^* T_u T_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to a  $*$ -homomorphism from  $\mathcal{D}_X$  to  $X$ ,

and thus that there exists a  $*$ -homomorphism from  $\mathcal{D}_{X_A} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $X$  sending  $S_u$  to  $T_u$  for every  $u \in \{1, 2, \dots, n\}^*$ , and especially  $S_i$  to  $T_i$  for every  $i \in \{1, 2, \dots, n\}$ .

It is clear from the way we defined  $T_u$  that condition (1) is satisfied. Let  $m \in \mathbb{N}$ , and denote by  $\mathcal{D}_m$  the  $C^*$ -subalgebra of  $\mathcal{D}_{X_A}$  generated by  $\{1_{C(u)}\}_{u \in \{1, 2, \dots, n\}^m}$ . If  $u, v \in \{1, 2, \dots, n\}^m$  and  $u \neq v$ , then

$$T_u T_u^* + T_v T_v^* \leq \sum_{w \in \{1, 2, \dots, n\}^m} T_w T_w^* = I,$$

and so

$$T_u^* T_u + T_u^* T_v T_v^* T_u = T_u^* (T_u T_u^* + T_v T_v^*) T_u \leq T_u^* I T_u = T_u^* T_u,$$

which implies that  $T_u T_u^* T_v T_v^* = T_u T_u^* T_v T_v^* T_u T_u^* = 0$ .

Thus  $\{T_u T_u^*\}_{u \in \{1, 2, \dots, n\}^m}$  are mutual orthogonal projections, and since  $\{1_{C(u)}\}_{u \in \{1, 2, \dots, n\}^m}$  also are mutual orthogonal projections and

$$\begin{aligned} 1_{C(u)} = 0 &\Rightarrow C(u) = \emptyset \\ &\Rightarrow u \notin \mathcal{L}(X_A) \\ &\Rightarrow \exists i \in \{1, 2, \dots, m-1\} : A(u_i, u_{i+1}) = 0 \\ &\Rightarrow T_{u_i} T_{u_{i+1}} = T_{u_i} T_{u_i}^* T_{u_i} T_{u_{i+1}} T_{u_{i+1}}^* T_{u_{i+1}} = \\ &\quad T_{u_i} \sum_{k=1}^n A(u_i, k) T_k T_k^* T_{u_{i+1}} T_{u_{i+1}}^* T_{u_{i+1}} = 0 \\ &\Rightarrow T_u T_u^* = 0, \end{aligned}$$

there is a unital  $*$ -homomorphism  $\psi_m$  from  $\mathcal{D}_m$  to  $X$  obeying  $\psi_m(1_{C(u)}) = T_u T_u^*$  for every  $u \in \{1, 2, \dots, n\}^m$ .

Since  $C(u)$  is the disjoint union of  $\{C(ui)\}_{i \in \{1, 2, \dots, n\}}$ ,

$$1_{C(u)} = \sum_{i=1}^n 1_{C(ui)} \in \mathcal{D}_{m+1}$$

for every  $u \in \{1, 2, \dots, n\}^m$ , so  $\mathcal{D}_m \subseteq \mathcal{D}_{m+1}$ . Let us denote the inclusion of  $\mathcal{D}_m$  into  $\mathcal{D}_{m+1}$  by  $\iota_m$ . Since

$$\begin{aligned} \psi_{m+1}(1_{C(u)}) &= \psi_{m+1}\left(\sum_{i=1}^n 1_{C(ui)}\right) \\ &= \sum_{i=1}^n T_{ui} T_{ui}^* \\ &= T_u \left(\sum_{i=1}^n T_i T_i^*\right) T_u^* \\ &= T_u T_u^* = \psi_m(1_{C(u)}), \end{aligned}$$

$\psi_{m+1} \circ \iota_m = \psi_m$ . Thus the  $\psi_m$ 's extends to a  $*$ -homomorphism from  $\overline{\bigcup_{m \in \mathbb{N}} \mathcal{D}_m}$  to  $X$ . It is easy to check that

$$1_{C(u, \epsilon)} = \begin{cases} \sum_{j=1}^n A(u_{|u|}, j) 1_{C(j)} & \text{if } u \in \mathbf{L}(\mathbf{X}_A) \\ 0 & \text{if } u \notin \mathbf{L}(\mathbf{X}_A), \end{cases}$$

and

$$1_{C(u, v)} = \begin{cases} 1_{C(v)} & \text{if } A(u_1, u_2) = A(u_2, u_3) = \dots = A(u_{|u|-1}, u_{|u|}) = \\ & = A(u_{|u|}, v_1) = 1, \\ 0 & \text{else,} \end{cases}$$

if  $v \neq \epsilon$ , and that

$$T_u^* T_u = \begin{cases} \sum_{j=1}^n A(u_{|u|}, j) T_j T_j^* & \text{if } u \in \mathbf{L}(\mathbf{X}_A) \\ 0 & \text{if } u \notin \mathbf{L}(\mathbf{X}_A), \end{cases}$$

and

$$T_v T_u^* T_u T_v^* = \begin{cases} T_v T_v^* & \text{if } A(u_1, u_2) = A(u_2, u_3) = \dots = A(u_{|u|-1}, u_{|u|}) = \\ & = A(u_{|u|}, v_1) = 1, \\ 0 & \text{else.} \end{cases}$$

So  $\mathcal{D}_{\mathbf{X}_A}$  is contained in  $\overline{\bigcup_{m \in \mathbb{N}} \mathcal{D}_m}$ , and  $\psi(1_{C(u, v)}) = T_v T_u^* T_u T_v^*$  for all  $u, v \in \{1, 2, \dots, n\}^*$ .  $\square$

This result is generalized in [4], where it is shown that  $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is isomorphic to a universal Cuntz-Krieger algebra, when  $\mathbf{X}$  is a sofic shift.

If  $A(i, j) = 1$  for every  $i, j \in \{1, 2, \dots, n\}$ , then  $\mathcal{O}_A$ , and hence  $\mathcal{D}_{\mathbf{X}_A} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , is the Cuntz algebra  $\mathcal{O}_n$  which was originally defined in [14]. The Cuntz algebras have proved to be very important examples in the theory of  $C^*$ -algebras, for example in classification of  $C^*$ -algebras (see [39]), and in the study of wavelets (see [2]).

## 9. UNIQUENESS AND A FAITHFUL REPRESENTATION

It follows from the universal property of  $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  that there exists an action  $\gamma : \mathbb{T} \rightarrow \text{Aut}(\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N})$  defined by  $\gamma_z(S_u) = z^{|u|} S_u$  for every  $z \in \mathbb{T}$ . This action is known as the *gauge action*.

Let  $\mathcal{F}_X$  denote the  $C^*$ -subalgebra of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  generated by  $\{S_v S_u^* S_w^* \mid u, v, w \in \mathfrak{a}^*, |v| = |w|\}$ . It is not difficult to see that

$$\left\{ \sum_{v \in J_-} X_v S_v^* + X_0 + \sum_{u \in J_+} S_u X_u \mid J_- \text{ and } J_+ \text{ are finite subset of } \mathfrak{a}^* \right. \\ \left. \text{and } X_0, X_v, X_u \in \mathcal{F}_X \text{ for all } v \in J_-, u \in J_+ \right\}$$

is a dense  $*$ -subalgebra  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ . Thus we see that  $\mathcal{F}_X$  is the fix point algebra of the gauge action.

If we let

$$E(X) = \int_{\mathbb{T}} \alpha_z(X) dz$$

for every  $X \in \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , then  $E$  is a projection of norm one from  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  onto  $\mathcal{F}_X$  satisfying

$$(11) \quad E(abc) = aE(b)c \text{ for all } a, c \in \mathcal{F}_X, \quad (\text{module property})$$

$$(12) \quad E(b^*b) \geq 0, \quad (\text{positivity})$$

$$(13) \quad E(b^*b) = 0 \text{ implies that } b = 0. \quad (\text{faithfulness})$$

for all  $b \in \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ . Thus

$$E\left(\sum_{v \in J_-} X_v S_v^* + X_0 + \sum_{u \in J_+} S_u X_u\right) = X_0$$

for all finite subset  $J_-, J_+$  of  $\mathfrak{a}^*$  and  $X_0, X_v, X_u \in \mathcal{F}_X$ ,  $v \in J_-, u \in J_+$ .

Building on the work done by Matsumoto in [23], the following Theorem is proved in [7]:

**Theorem 13.** *Let  $X$  be a one-sided shift space,  $X$  is a  $C^*$ -algebra generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$ , and  $\phi : \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \rightarrow X$  a  $*$ -homomorphism such that  $\phi(S_u) = s_u$  for every  $u \in \mathfrak{a}^*$ . Then the following three statements are equivalent:*

- (1) *the  $*$ -homomorphism  $\phi : \mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \rightarrow X$  is injective,*
- (2) *the restriction of  $\phi$  to  $\mathcal{A}_X$  is injective and there exists an action  $\gamma : \mathbb{T} \rightarrow \text{Aut}(X)$  such that  $\gamma_z(s_u) = z^{|u|} s_u$  for every  $z \in \mathbb{T}$  and every  $u \in \mathfrak{a}^*$ ,*
- (3) *the restriction of  $\phi$  to  $\mathcal{A}_X$  is injective and there exists a projection  $E$  of norm one from  $X$  onto  $C^*(s_v s_u^* s_w^* \mid u, v, w \in \mathfrak{a}^*, |v| = |w|)$  satisfying*

$$E(abc) = aE(b)c \text{ for all } a, c \in C^*(s_v s_u^* s_w^* \mid u, v, w \in \mathfrak{a}^*, |v| = |w|),$$

$$E(b^*b) \geq 0,$$

$$E(b^*b) = 0 \text{ implies that } b = 0,$$

for all  $b \in X$ .

As a corollary to this theorem we get:

**Corollary 14.** *Let  $X$  be a one-sided shift space. If  $X$  is a  $C^*$ -algebra generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfying:*

- (1)  $s_u s_v = s_{uv}$  for all  $u, v \in \mathfrak{a}^*$ ,
- (2) the map

$$1_{C(u,v)} \mapsto s_v s_u^* s_u s_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to an injective  $*$ -homomorphism from  $\mathcal{D}_X$  to  $X$ ,

(3) there exists an action  $\gamma : \mathbb{T} \rightarrow \text{Aut}(X)$  defined by  $\gamma_z(s_u) = z^{|u|}s_u$  for every  $z \in \mathbb{T}$ , then  $X$  and  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  are isomorphic by an isomorphism which maps  $s_u$  to  $S_u$  for every  $u \in \mathfrak{a}^*$ .

As a consequence of this, we are now able to construct for every one-sided shift space  $X$  a faithful representation of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  in the following way. Let  $\mathfrak{H}_X$  be the Hilbert space  $l_2(X) \oplus l_2(\mathbb{Z})$  with orthonormal basis  $(e_x, e_n)_{x \in X, n \in \mathbb{Z}}$ , and let for every  $u \in \mathfrak{a}^*$ ,  $s_u$  be the operator on  $\mathfrak{H}_X$  defined by:

$$s_u(e_x, e_n) = \begin{cases} (e_{ux}, e_{n+|u|}) & \text{if } ux \in X, \\ 0 & \text{if } ux \notin X. \end{cases}$$

It is easy to check that  $s_us_v = s_{uv}$  and that

$$s_vs_u^*s_us_v^*(e_x, e_n) = \begin{cases} (e_x, e_n) & \text{if } x \in C(u, v), \\ 0 & \text{if } x \notin C(u, v). \end{cases}$$

Thus  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfies (1) and (2) of Corollary 14.

If we for every  $z \in \mathbb{T}$  let  $U_z$  be the operator on  $\mathfrak{H}_X$  defined by

$$U_z(e_x, e_n) = z^n(e_x, e_n),$$

then  $U_z$  is a unitary operator on  $\mathfrak{H}_X$ , and  $U_z s_u U_z^* = z^{|u|}s_u$  for every  $u \in \mathfrak{a}^*$ . Thus  $\{s_u\}_{u \in \mathfrak{a}^*}$  also satisfies (3) of Corollary 14, and therefore  $S_u \mapsto s_u$  is a faithful representation of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ .

**Definition 15.** We say that a one-sided shift space  $X$  satisfies condition (I) if there for every  $x \in X$  and every  $l \in \mathbb{N}_0$  exists a  $y \in X$  such that  $\mathcal{P}_l(x) = \mathcal{P}_l(y)$  and  $x \neq y$ .

One can show that if  $X$  satisfies condition (I), then there for all  $C^*$ -algebra  $X$  generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfying:

- (1)  $s_us_v = s_{uv}$  for all  $u, v \in \mathfrak{a}^*$ ,
- (2) the map

$$1_{C(u,v)} \mapsto s_vs_u^*s_us_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to an injective  $*$ -homomorphism from  $\mathcal{D}_X$  to  $X$ ,

exists an action  $\gamma : \mathbb{T} \rightarrow \text{Aut}(X)$  such that  $\gamma_z(s_u) = z^{|u|}s_u$  for every  $z \in \mathbb{T}$ .

This was first proved by Matsumoto in the case where  $X$  is of the form  $X_\Lambda$  for some two-sided shift space  $\Lambda$  in [25], where he also discuss several conditions which are equivalent of condition (I), and this has been generalized to arbitrary one-sided shift spaces  $X$  by the first author in [3].

From this result follows the following theorem:

**Theorem 16.** Let  $X$  be a one-sided shift space which satisfies condition (I). If  $X$  is a  $C^*$ -algebra generated by partial isometries  $\{s_u\}_{u \in \mathfrak{a}^*}$  satisfying:

- (1)  $s_us_v = s_{uv}$  for all  $u, v \in \mathfrak{a}^*$ ,
- (2) the map

$$1_{C(u,v)} \mapsto s_vs_u^*s_us_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to an injective  $*$ -homomorphism from  $\mathcal{D}_X$  to  $X$ ,

then  $X$  and  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  are isomorphic by an isomorphism which maps  $s_u$  to  $S_u$  for every  $u \in \mathfrak{a}^*$ .

10. PROPERTIES OF  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ 

We will in this section shortly describe some of the properties of the  $C^*$ -algebra  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ .

As mentioned in Remark 9,  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is isomorphic to the  $C^*$ -algebra  $\mathcal{O}_X$  defined in [5], and since  $\mathcal{O}_X$  is the  $C^*$ -algebra of a separable  $C^*$ -correspondence over  $\mathcal{D}_X$  which is separable and commutative and hence nuclear and satisfies the UCT, the same is the case for the  $C^*$ -algebra  $J_X$  mentioned in [21, Proposition 8.8], and thus it follows from [21, Corollary 7.4 and Proposition 8.8] that  $\mathcal{O}_X$  and hence  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is nuclear and satisfies the UCT.

**Theorem 17.** *Let  $X$  be a one-sided shift space. Then the  $C^*$ -algebra  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is nuclear and satisfies the UCT.*

Matsumoto has in [25] proved the following:

**Theorem 18.** *Let  $\Lambda$  be a two-sided shift space. We then have:*

- (1) *if  $X_\Lambda$  is irreducible in past equivalence, meaning that there for every  $l \in \mathbb{N}_0$ , every  $y \in X_\Lambda$  and every sequence  $(x_n)_{n \in \mathbb{N}}$  of  $X_\Lambda$  such that  $\mathcal{P}_n(x_n) = \mathcal{P}_n(x_{n+1})$  for every  $n \in \mathbb{N}$ , exist  $N \in \mathbb{N}$  and a  $u \in \mathcal{L}(\Lambda)$  such that  $\mathcal{P}_l(y) = \mathcal{P}_l(ux_{l+N})$ , then the  $C^*$ -algebra  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is simple;*
- (2) *if  $X_\Lambda$  is aperiodic in past equivalence, meaning that there for any  $l \in \mathbb{N}_0$  exists  $N \in \mathbb{N}$  such that for any pair  $x, y \in X_\Lambda$ , exists  $u \in \mathcal{L}_N(\Lambda)$  such that  $\mathcal{P}_l(y) = \mathcal{P}_l(ux)$ , then the  $C^*$ -algebra  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is simple and purely infinite.*

11.  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  AS AN INVARIANT

We will in this section see that  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is an invariant for one-sided conjugacy in the sense that if two one-sided shift spaces  $X$  and  $Y$  are conjugate, then  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  are isomorphic.

This was first proved by Matsumoto in [23] for the special case where  $X = X_\Lambda$  and  $Y = X_\Gamma$  for two two-sided shift spaces  $\Lambda$  and  $\Gamma$  satisfying condition (I), and generalized to the general case in [5]. Because of the way we have constructed  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  in this paper we can very easily prove this result and even improve it a little bit.

Remember that in  $\mathcal{T}(\mathcal{D}_X, \alpha, \mathcal{L})$ ,  $S^*aS = \mathcal{L}(a)$  for every  $a \in \mathcal{D}_X$ , so in  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$   $\rho(S)^*a\rho(S) = \mathcal{L}(a)$  for every  $a \in \mathcal{D}_X$ . We will therefore denote the map

$$a \mapsto \rho(S)^*a\rho(S)$$

from  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  by  $\mathcal{L}$ . We will by  $\lambda_X$  denote the map

$$X \mapsto \left( \sum_{a \in \mathfrak{a}} S_a^* \right) X \left( \sum_{b \in \mathfrak{a}} S_b \right)$$

from  $\mathcal{F}_X$  to  $\mathcal{F}_X$ .

**Theorem 19.** *If  $X$  and  $Y$  are two one-sided shift spaces which are conjugate, then there exists a  $*$ -isomorphism  $\Phi$  from  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  such that:*

- (1)  $\Phi(C(X)) = C(Y)$ ,
- (2)  $\Phi(\mathcal{D}_X) = \mathcal{D}_Y$ ,
- (3)  $\Phi(\mathcal{F}_X) = \mathcal{F}_Y$ ,
- (4)  $\Phi \circ \alpha_X = \alpha_Y$ ,

- (5)  $\Phi \circ \gamma_z = \gamma_z$  for every  $z \in \mathbb{T}$ ,
- (6)  $\Phi \circ \mathcal{L}_X = \mathcal{L}_Y$ ,
- (7)  $\Phi \circ \lambda_X = \lambda_Y$ .

*Proof.* Let  $\phi$  be a conjugacy between  $Y$  and  $X$ , and let  $\Phi$  be the map between the bounded functions on  $X$  and the bounded functions on  $Y$  defined by

$$f \mapsto f \circ \phi.$$

Then  $\Phi(C(X)) = C(Y)$ ,  $\Phi \circ \alpha_X = \alpha_Y \circ \Phi$  and  $\Phi \circ \mathcal{L}_Y = \mathcal{L}_X \circ \Phi$ , and hence  $\Phi(\mathcal{D}_Y) = \mathcal{D}_X$ . Thus it follows from the construction of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  that there is a  $*$ -isomorphism from  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  which extends  $\Phi$ , maps  $\rho(S)$  to  $\rho(S)$  and satisfies  $\Phi \circ \alpha_X = \alpha_Y$ . We will also denote this  $*$ -isomorphism by  $\Phi$ .

Since the gauge action of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is characterized by  $\gamma_z(f) = f$  for all  $f \in \mathcal{D}_X$  and  $\gamma_z(\rho(S)) = z\rho(S)$  and the gauge action of  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is characterized in the same way, we see that  $\Phi \circ \gamma_z = \gamma_z$  for every  $z \in \mathbb{T}$ .

Since  $\mathcal{F}_X$  is the fix point algebra of the gauge action of  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and  $\mathcal{F}_Y$  is the fix point algebra of the gauge action of  $\mathcal{D}_Y \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , we have that  $\Phi(\mathcal{F}_X) = \mathcal{F}_Y$ .

Since  $\Phi$  maps  $\rho(S)$  to  $\rho(S)$ , we have that  $\Phi \circ \mathcal{L}_X = \mathcal{L}_Y$ .

Let us denote the function

$$x \mapsto \sigma^{-1}\{x\}, \quad x \in X$$

by  $f_X$  and the function

$$x \mapsto \sigma^{-1}\{x\}, \quad x \in Y$$

by  $f_Y$ . We then have that

$$\lambda_X(X) = \left( \sum_{a \in \mathfrak{a}} S_a^* \right) X \left( \sum_{b \in \mathfrak{a}} S_b \right) = \rho(S)^* \alpha(f_X)^{1/2} X \alpha(f_X)^{1/2} \rho(S),$$

and since  $\Phi(f_X) = f_Y$ , we have that  $\Phi \circ \lambda_X = \lambda_Y$ .  $\square$

If two two-sided shift spaces  $\Lambda$  and  $\Gamma$  are flow equivalent, then the corresponding one-sided shift spaces  $X_\Lambda$  and  $X_\Gamma$  are not necessarily conjugate, so we cannot expect that  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and  $\mathcal{D}_{X_\Gamma} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  are isomorphic (and there are examples of two two-sided shift spaces  $\Lambda$  and  $\Gamma$ , such that  $\Lambda$  and  $\Gamma$  are conjugate and hence flow equivalent, but  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  and  $\mathcal{D}_{X_\Gamma} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  are not isomorphic), but it turns out that  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \otimes \mathcal{K}$  and  $\mathcal{D}_{X_\Gamma} \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators on a separable Hilbert space. This has been proved by Matsumoto in [29] for  $\Lambda$  and  $\Gamma$  satisfying condition (I), and in generality in [7].

## 12. THE $K$ -THEORY OF $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$

Since  $K_0$  and  $K_1$  are invariants of a  $C^*$ -algebra, it follows from the previous section that  $K_0(\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N})$ ,  $K_1(\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N})$  and  $K_0(\mathcal{F}_X)$  are invariants of  $X$ . We will in this section present formulas based on  $l$ -past equivalence for these invariants. This was done in [24, 25, 32] for the case of one-sided shift spaces of the form  $X_\Lambda$ , where  $\Lambda$  is a two-sided shift space and generalized to the general case in [3].

One can directly from these formulas prove that there are invariants of  $X$  without involving  $C^*$ -algebras. This is done (for one-sided shift spaces of the form  $X_\Lambda$ , where  $\Lambda$  is a two-sided shift space) in Matsumoto's outstanding paper [26], where also other invariants of shift spaces are presented.

Let  $\mathbf{X}$  be a one-sided shift space. We let for each  $l \in \mathbb{N}_0$ ,  $m(l)$  be the number of  $l$ -past equivalence classes and we denote the  $l$ -past equivalence classes by  $\mathcal{E}_1^l, \mathcal{E}_2^l, \dots, \mathcal{E}_{m(l)}^l$ . For each  $l \in \mathbb{N}_0$ ,  $j \in \{1, 2, \dots, m(l)\}$  and  $i \in \{1, 2, \dots, m(l+1)\}$  we let

$$I_l(i, j) = \begin{cases} 1 & \text{if } \mathcal{E}_i^{l+1} \subseteq \mathcal{E}_j^l \\ 0 & \text{else.} \end{cases}$$

Let  $F$  be a finite set and  $i_0 \in F$ . Then we denote by  $e_{i_0}$  the element in  $\mathbb{Z}^F$  for which

$$e_{i_0}(i) = \begin{cases} 1 & \text{if } i = i_0 \\ 0 & \text{else.} \end{cases}$$

For  $0 \leq k \leq l$  let  $M_k^l = \{i \in \{1, 2, \dots, m(l)\} \mid \mathcal{P}_k(\mathcal{E}_i^l) \neq \emptyset\}$ . Since  $i \in M_k^{l+1}$  if  $j \in M_k^l$  and  $I_l(i, j) = 1$ , there exists a positive, linear map from  $\mathbb{Z}^{M_k^l}$  to  $\mathbb{Z}^{M_k^{l+1}}$  given by

$$e_j \mapsto \sum_{i \in M_k^{l+1}} I_l(i, j) e_i.$$

We denotes this map by  $I_k^l$ .

For a subset  $\mathcal{E}$  of  $\mathbf{X}$  and a  $u \in \mathfrak{a}^*$  we let  $u\mathcal{E} = \{ux \in \mathbf{X} \mid x \in \mathcal{E}\}$ . For each  $l \in \mathbb{N}_0$ ,  $j \in \{1, 2, \dots, m(l)\}$ ,  $i \in \{1, 2, \dots, m(l+1)\}$  and  $a \in \mathfrak{a}$  we let

$$A_l(i, j, a) = \begin{cases} 1 & \text{if } \emptyset \neq a\mathcal{E}_i^{l+1} \subseteq \mathcal{E}_j^l \\ 0 & \text{else.} \end{cases}$$

Let  $0 \leq k \leq l$ . If  $j \in M_k^l$  and there exists an  $a \in \mathfrak{a}$  such that  $A_l(i, j, a) = 1$ , then  $i \in M_{k+1}^{l+1}$ . Hence there exists a positive, linear map from  $\mathbb{Z}^{M_k^l}$  to  $\mathbb{Z}^{M_{k+1}^{l+1}}$  given by

$$e_j \mapsto \sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} A_l(i, j, a) e_i.$$

We denote this map by  $A_k^l$ .

Then we have:

**Lemma 20.** *Let  $0 \leq k \leq l$ . Then the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}^{M_k^l} & \xrightarrow{I_k^l} & \mathbb{Z}^{M_k^{l+1}} \\ A_k^l \downarrow & & \downarrow A_k^{l+1} \\ \mathbb{Z}^{M_{k+1}^{l+1}} & \xrightarrow{I_{k+1}^{l+1}} & \mathbb{Z}^{M_{k+1}^{l+2}} \end{array}$$

*Proof.* Let  $j \in M_k^l$ ,  $h \in M_{k+1}^{l+2}$  and  $a \in \mathfrak{a}$ . If  $\emptyset \neq a\mathcal{E}_h^{l+2} \subseteq \mathcal{E}_j^l$ , then there exists exactly one  $i \in M_k^{l+1}$  such that  $\mathcal{E}_i^{l+1} \subseteq \mathcal{E}_j^l$  and  $\emptyset \neq a\mathcal{E}_h^{l+2} \subseteq \mathcal{E}_i^{l+1}$ ; and there exists exactly one  $i' \in M_{k+1}^{l+1}$  such that  $\mathcal{E}_h^{l+2} \subseteq \mathcal{E}_{i'}^{l+1}$  and  $\emptyset \neq a\mathcal{E}_{i'}^{l+1} \subseteq \mathcal{E}_j^l$ ; and if  $a\mathcal{E}_h^{l+2} = \emptyset$  or  $a\mathcal{E}_h^{l+2} \not\subseteq \mathcal{E}_j^l$  then there does not exists a  $i \in M_k^{l+1}$  such that  $\mathcal{E}_i^{l+1} \subseteq \mathcal{E}_j^l$  and  $\emptyset \neq a\mathcal{E}_h^{l+2} \subseteq \mathcal{E}_i^{l+1}$ ; and there does not exists a  $i' \in M_{k+1}^{l+1}$  such that  $\mathcal{E}_h^{l+2} \subseteq \mathcal{E}_{i'}^{l+1}$  and  $\emptyset \neq a\mathcal{E}_{i'}^{l+1} \subseteq \mathcal{E}_j^l$ . Hence

$$\sum_{i \in M_k^{l+1}} A_{l+1}(h, i, a) I_l(i, j) = \sum_{i \in M_{k+1}^{l+1}} I_{l+1}(h, i) A_l(i, j, a).$$



So

$$\begin{aligned}
A_k^{l+1}(I_k^l(e_j)) &= A_k^{l+1} \left( \sum_{i \in M_k^{l+1}} I_l(i, j) e_i \right) \\
&= \sum_{h \in M_{k+1}^{l+2}} \sum_{a \in \mathfrak{a}} A_{l+1}(h, i, a) \sum_{i \in M_k^{l+1}} I_l(i, j) e_h \\
&= \sum_{h \in M_{k+1}^{l+2}} \sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} I_{l+1}(h, i) A_l(i, j, a) e_h \\
&= I_{k+1}^{l+1} \left( \sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} A_l(i, j, a) e_i \right) \\
&= I_{k+1}^{l+1}(A_k^l(e_j))
\end{aligned}$$

for every  $j \in M_k^l$ . Hence the diagram commutes.  $\square$

For  $k \in \mathbb{N}_0$  the inductive limit  $\varinjlim (\mathbb{Z}^{M_k^l}, (\mathbb{Z}^+)^{M_k^l}, I_k^l)$  will be denoted by  $(\mathbb{Z}_{\mathbf{x}_k}, \mathbb{Z}_{\mathbf{x}_k}^+)$ . It follows from Lemma 20 that the  $A_k^l$ 's induce a positive, linear map  $A_k$  from  $\mathbb{Z}_{\mathbf{x}_k}$  to  $\mathbb{Z}_{\mathbf{x}_{k+1}}$ .

Let  $0 \leq k < l$ . Denote by  $\delta_k^l$  the linear map from  $\mathbb{Z}^{M_k^l}$  to  $\mathbb{Z}^{M_{k+1}^l}$  given by

$$e_j \mapsto \begin{cases} e_j & \text{if } j \in M_{k+1}^l, \\ 0 & \text{if } j \notin M_{k+1}^l, \end{cases}$$

for  $j \in M_k^l$ . It is easy to check that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{Z}^{M_k^l} & \xrightarrow{\delta_k^l} & \mathbb{Z}^{M_{k+1}^l} \\
I_k^l \downarrow & & \downarrow I_{k+1}^l \\
\mathbb{Z}^{M_k^{l+1}} & \xrightarrow{\delta_k^{l+1}} & \mathbb{Z}^{M_{k+1}^{l+1}}
\end{array}$$

Thus the  $\delta_k^l$ 's induce a positive, linear map from  $\mathbb{Z}_{\mathbf{x}_k}$  to  $\mathbb{Z}_{\mathbf{x}_{k+1}}$  which we denote by  $\delta_k$ . Since the diagram

$$\begin{array}{ccc}
\mathbb{Z}^{M_k^l} & \xrightarrow{\delta_k^l} & \mathbb{Z}^{M_{k+1}^l} \\
A_k^l \downarrow & & \downarrow A_{k+1}^l \\
\mathbb{Z}^{M_{k+1}^{l+1}} & \xrightarrow{\delta_{k+1}^{l+1}} & \mathbb{Z}^{M_{k+2}^{l+1}}
\end{array}$$

commutes for every  $0 \leq k < l$ , the diagram

$$\begin{array}{ccc}
\mathbb{Z}_{\mathbf{x}_k} & \xrightarrow{\delta_k} & \mathbb{Z}_{\mathbf{x}_{k+1}} \\
A_k \downarrow & & \downarrow A_{k+1} \\
\mathbb{Z}_{\mathbf{x}_{k+1}} & \xrightarrow{\delta_{k+1}} & \mathbb{Z}_{\mathbf{x}_{k+2}}
\end{array}$$

commutes.

We denote the inductive limit  $\varinjlim (\mathbb{Z}_{\mathbf{X}_k}, \mathbb{Z}_{\mathbf{X}_k}^+, A_k)$  by  $(\Delta_{\mathbf{X}}, \Delta_{\mathbf{X}}^+)$ . Since the previous diagram commutes, the  $\delta_k$ 's induce a positive, linear map from  $\Delta_{\mathbf{X}}$  to  $\Delta_{\mathbf{X}}$  which we denote by  $\delta_{\mathbf{X}}$ .

**Theorem 21.** *For every one-sided shift space  $\mathbf{X}$  is*

$$(K_0(\mathcal{F}_{\mathbf{X}}), K_0^+(\mathcal{F}_{\mathbf{X}}), (\lambda_{\mathbf{X}})_*) \cong (\Delta_{\mathbf{X}}, \Delta_{\mathbf{X}}^+, \delta_{\mathbf{X}}),$$

or more precisely, the map  $[S_u 1_{\mathcal{E}_i^l} S_v^*]_0 \mapsto e_i \in \mathbb{Z}^{M_k^l}$  extends to an isomorphism from  $(K_0(\mathcal{F}_{\mathbf{X}}), K_0^+(\mathcal{F}_{\mathbf{X}}), (\lambda_{\mathbf{X}})_*)$  to  $(\Delta_{\mathbf{X}}, \Delta_{\mathbf{X}}^+, \delta_{\mathbf{X}})$ .

Denote for every  $l \in \mathbb{N}_0$  by  $B^l$  the linear map from  $\mathbb{Z}^{M_1^l}$  to  $\mathbb{Z}^{m(l+1)}$  given by

$$e_j \mapsto \sum_{i=1}^{m(l+1)} \left( I_l(i, j) - \sum_{a \in \mathfrak{a}} A_l(i, j, a) \right) e_i.$$

One can easily check that the following diagram commutes for every  $l \in \mathbb{N}_0$ .

$$\begin{array}{ccc} \mathbb{Z}^{M_1^l} & \xrightarrow{B^l} & \mathbb{Z}^{m(l+1)} \\ I_1^l \downarrow & & \downarrow I_0^{l+1} \\ \mathbb{Z}^{M_1^{l+1}} & \xrightarrow{B^{l+1}} & \mathbb{Z}^{m(l+2)}. \end{array}$$

Hence the  $B^l$ 's induce a linear map  $B$  from  $\mathbb{Z}_{\Lambda_1}$  to  $\mathbb{Z}_{\Lambda_0}$ .

**Theorem 22.** *Let  $\Lambda$  be a one-sided shift space. Then*

$$K_0(\mathcal{O}_{\Lambda}) \cong \mathbb{Z}_{\Lambda_0} / B\mathbb{Z}_{\Lambda_1},$$

and

$$K_1(\mathcal{O}_{\Lambda}) \cong \ker(B).$$

More precisely: The map

$$[1_{\mathcal{E}_i^l}]_0 \mapsto e_i \in \mathbb{Z}^{m(l)}$$

induces an isomorphism from  $K_0(\mathcal{O}_{\Lambda})$  to  $\mathbb{Z}_{\Lambda_0} / B\mathbb{Z}_{\Lambda_1}$ .

### 13. THE IDEAL STRUCTURE OF $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$

We will in this section describe the structure of the gauge invariant ideals of  $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ . By an ideal we will in this paper always mean a closed two-sided ideal, and by a gauge invariant ideal, we mean an ideal  $I$  such that  $\gamma_z(I) \subseteq I$  for every  $z \in \mathbb{T}$ .

The lattice of the gauge invariant ideals of  $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  has been described by Matsumoto in [25] in the case where  $\mathbf{X}$  is of the form  $\mathbf{X}_{\Lambda}$  for some two-sided shift space  $\Lambda$  and this has been generalized to arbitrary one-sided shift spaces  $\mathbf{X}$  by the first author in [3]. We will here reformulate the description a bit.

**Theorem 23.** *Let  $\mathbf{X}$  be a one-sided shift space. Then there exist between each pair of the following lattices an ordering preserving bijective map:*

- (1) the lattice of gauge invariant ideals of  $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ ,
- (2) the lattice of ideals  $J$  of  $\mathcal{F}_{\mathbf{X}}$ , such that  $S_u X S_u^*, S_u^* X S_u \in J$  for every  $u \in \mathfrak{a}^*$  and every  $X \in J$ ,
- (3) the lattice of ideals  $I$  of  $\mathcal{A}_{\mathbf{X}}$ , such that  $S_u^* X S_u \in I$  for every  $u \in \mathfrak{a}^*$  and every  $X \in I$ ,

- (4) the lattice of order ideals of  $\Delta_X$  invariant under  $\delta_X$ ,
- (5) the lattice of subset  $A$  of  $X$ , such that  $\sigma(A) \subseteq A$  and  $\forall x \in A \exists l \in \mathbb{N}_0 : \mathcal{P}_l(x) \subseteq A$ .

#### 14. EXAMPLES

If  $\Lambda$  is a two-sided shift space, then as explained before we can associate to it the  $C^*$ -algebra  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , but we of course also look at the  $C^*$ -crossed product  $C(\Lambda) \rtimes_{\phi} \mathbb{Z}$ , where  $\phi : C(\Lambda) \rightarrow C(\Lambda)$  is the map

$$f \mapsto f \circ \sigma.$$

It is proved in [6] that if  $\Lambda$  satisfy the condition

$$(*) : \text{There exists for every } u \in \mathcal{L}(\Lambda) \text{ an } x \in X_\Lambda \text{ such that } \mathcal{P}_{|u|}(x) = \{u\},$$

then  $C(\Lambda) \rtimes_{\phi} \mathbb{Z}$  is a quotient of  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ . This is used in [11] and [12] to relate the  $K$ -theory of  $\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  to the  $K$ -theory of  $C(\Lambda) \rtimes_{\phi} \mathbb{Z}$  for these shift spaces, and in [10] to present  $K_0(\mathcal{D}_{X_\Lambda} \rtimes_{\alpha, \mathcal{L}} \mathbb{N})$ , for a two-sided shift space  $\Lambda$  associated to an aperiodic and primitive substitution, as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [8] and [9]).

In [34], Matsumoto has taken a closer look at  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  in the case where  $X$  is the Motzkin shift, and in [27] he examines  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  for the context-free shift. In [22]  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is examined for the Dyck shift, and in [19]  $\mathcal{D}_X \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is examined for a class of shift spaces called  $\beta$ -shifts.

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